More Angular Momentum

Physics 1425 Lecture 22
Torque as a Vector

• Suppose we have a wheel spinning about a fixed axis: then $\vec{\omega}$ always points along the axis—so $d\vec{\omega}/dt$ points along the axis too.

• If we want to write a vector equation

$$\vec{\tau} = I\vec{\alpha} = Id\vec{\omega}/dt$$

it’s clear that the vector $\vec{\tau}$ is parallel to the vector $d\vec{\omega}/dt$: so $\vec{\tau}$ points along the axis too!

• **BUT** this vector $\vec{\tau}$, is, remember made of two other vectors: the force $\vec{F}$ and the place $\vec{r}$ where it acts!
More Torque...

• Expressing the force vector \( \vec{F} \) as a sum of components \( \vec{F}_\perp \) (“fperp”) perpendicular to the lever arm and \( \vec{F}_\parallel \) parallel to the arm, it’s clear that only \( \vec{F}_\perp \) has leverage, that is, torque, about O. \( \vec{F}_\perp \) has magnitude \( F \sin \theta \), so \( \tau = rF \sin \theta \).

• Alternatively, keep \( \vec{F} \) and measure its lever arm about O: that’s \( r \sin \theta \).
Definition: The Vector Cross Product

\[ \vec{C} = \vec{A} \times \vec{B} \]

- The magnitude \( C \) is \( AB\sin \theta \), where \( \theta \) is the angle between the vectors \( \vec{A}, \vec{B} \).
- The direction of \( \vec{C} \) is perpendicular to both \( \vec{A} \) and \( \vec{B} \), and is your right thumb direction if your curling fingers go from \( \vec{A} \) to \( \vec{B} \).
The Vector Cross Product in Components

- Recall we defined the unit vectors \( \hat{i}, \hat{j}, \hat{k} \) pointing along the x, y, z axes respectively, and a vector can be expressed as \( \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \).

- Now \( \hat{i} \times \hat{i} = 0, \hat{i} \times \hat{j} = \hat{k}, \hat{i} \times \hat{k} = -\hat{j}, \ldots \)

- So

\[
\vec{A} \times \vec{B} = \left( A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \right) \times \left( B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \right)
\]

\[
= \hat{i} \left( A_y B_z - A_z B_y \right) + \ldots
\]
Vector Angular Momentum of a Particle

- A particle with momentum $\vec{p}$ is at position $\vec{r}$ from the origin O.
- Its angular momentum about the origin is $\vec{L} = \vec{r} \times \vec{p}$.
- This is in line with our definition for part of a rigid body rotating about an axis: but also works for a particle flying through space.

Viewing the x-axis as coming out of the slide, this is a “right-handed” set of axes: $\hat{i} \times \hat{j} = +\hat{k}$
Angular Momentum and Torque for a Particle

- Angular momentum **about the origin** of particle mass $m$, momentum $\vec{p}$ at $\vec{r}$
  \[ \vec{L} = \vec{r} \times \vec{p} \]

- Rate of change:
  \[ \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \]

- because
  \[ \frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0. \]
Kepler’s Second Law

As the planet moves, a line from the planet to the center of the Sun sweeps out equal areas in equal times.

- In unit time, it moves through a distance $\vec{v}$.
- The area of the triangle swept out is $\frac{1}{2}rv\sin\theta$ (from $\frac{1}{2}$ base x height)
- This is $\frac{1}{2}L/m$, $\vec{L} = \vec{r} \times \vec{p}$.
- Kepler’s Law is telling us the angular momentum about the Sun is constant: this is because the Sun’s pull has zero torque about the Sun itself.

The base of the thin blue triangle is a distance $v$ along the tangent. The height is the perp distance of this tangent from the Sun.
Guy on Turntable

- A, of mass $m$, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius $R$, next to B, who’s on the ground.
- A now walks around the edge until he’s back with B.
- How far does he walk?

A. $2\pi R$
B. $2.5\pi R$
C. $3\pi R$
Guy on Turntable: Answer

- A, of mass $m$, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius $R$, next to B, who’s on the ground.
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  $$3\pi R$$

His moment of inertia is $mR^2$, the turntable’s is $2mR^2$. There is zero total angular momentum, so if he walks around with angular velocity $\omega$ relative to the ground, the turntable has angular velocity $-\omega/2$. If he marked the turntable at the point he began, he’d reach that mark again after walking 2/3rds of the way round, as the turntable turned the other way to meet him. When he gets back to B, the turntable has done half a complete turn.
Guy on Turntable Catches a Ball

- A, of mass $m$, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius $R$, at rest.
- B, who’s on the ground, throws a ball weighing $0.1m$ at speed $v$ to A, who catches it without slipping.
- What is the angular momentum of turntable + man + ball now?

A. $0.1mvR$
B. $(0.1/3.1)mvR$
C. $(0.1/5.1)mvR$
On the Ball? Answer

• A, of mass $m$, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius $R$, at rest.
• B, who’s on the ground, throws a ball weighing $0.1m$ at speed $v$ to A, who catches it without slipping.
• What is the angular momentum of turntable + man + ball now?

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The ball thrown from B to A is moving in the direction of the tangent at A, the angular momentum about a point of a particle flying through the air equals $\vec{r} \times m\vec{v}$ and the line of the velocity is perp to the radius ending at A, so the angular momentum of the ball about the disk center is $0.1mvR$. There is no other angular momentum, so this is shared with the man and the turntable.
Guy on Turntable Walks In

- A, of mass $m$, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius $R$, which is rotating at 6 rpm.
- A walks to the exact center of the turntable.
- How fast (approximately) is the turntable now rotating?

A. 12 rpm
B. 9 rpm
C. 6 rpm
D. 4 rpm
Guy on Turntable Walks In: Answer

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Initially, the man has moment of inertia $mR^2$, the turntable $2mR^2$. Finally, the man has negligible moment of inertia, so the total $I$ decreases by a factor of $2/3$, to conserve angular momentum (ther are no external torques) $\omega$ increases by $3/2$. 
Reminder: Angular Momentum and Torque for a Particle...

- Angular momentum about the origin of particle mass $m$, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$
Lots of Particles

• Suppose we have particles acted on by external forces, and also acting on each other.

• The rate of change of angular momentum of one of the particles about a fixed origin O is:

\[
\frac{d\mathbf{L}_i}{dt} = \tau_{i \text{ int}} + \tau_{i \text{ ext}}
\]

• The internal torques come in equal and opposite pairs, so

\[
\frac{d\mathbf{L}}{dt} = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \tau_{i \text{ ext}}
\]
Rotational Motion of a Rigid Body

• For a collection of interacting particles, we’ve seen that

\[ \frac{d\vec{L}}{dt} = \sum_i \vec{\tau}_i \]

the vector sum of the applied torques, \( \vec{L} \) and the \( \vec{\tau}_i \) being measured about a fixed origin O.

• A rigid body is equivalent to a set of connected particles, so the same equation holds.

• It is also true (proof in book) that even if the CM is accelerating,

\[ \frac{d\vec{L}_{CM}}{dt} = \sum \vec{\tau}_{CM} \]
Angular Velocity and Angular Momentum Need not be Parallel

• Imagine a dumbbell attached at its center of mass to a light vertical rod as shown, then the system rotates about the vertical line.

• The angular velocity vector $\vec{\omega}$ is vertical.

• The total angular momentum $\vec{L}$ about the CM is $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$.

• Think about this at the instant the balls are in the plane of the slide—so is $\vec{L}$, but it’s not vertical!
When *are* Angular Velocity and Angular Momentum Parallel?

- When the rotating object is symmetric about the axis of rotation: if for each mass on one side of the axis, there’s an equal mass at the corresponding point on the other side.
- For this pair of masses, \( \vec{r}_1 \times m\vec{\nu}_1 + \vec{r}_2 \times m\vec{\nu}_2 \) is along the axis.
- (Check it out!)