7. Time Evolution in Phase Space: Poisson Brackets and Constants of the Motion

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The Poisson Bracket

A function $f\left(p,q,t\right)$ of the phase space coordinates of the system and time has total time derivative

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\sum_i\left(\frac{\partial f}{\partial q_i}\dot{q}_i+\frac{\partial f}{\partial p_i}\dot{p}_i\right).$$

This is often written as

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\left[H,f\right]$$

where

$$\left[H,f\right]=\sum_i\left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right)$$

is called the *Poisson bracket*.

If, for a phase space function $f\left(p_i,q_i\right)$ (that is, no explicit time dependence) $\left[H,f\right]=0,$ then $f\left(p_i,q_i\right)$ is a constant of the motion, also called an *integral of the motion*.

In fact, the Poisson bracket can be defined for *any* two functions defined in phase space:

$$\left[f,g\right]=\sum_i\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right).$$

It's straightforward to check the following properties of the Poisson bracket:

$$\begin{aligned}
&\left[f,g\right]=-\left[g,f\right],\\
&\left[f,c\right]=0 \ \text{ for } c \text{ a constant,}\\
&\left[f_1+f_2,g\right]=\left[f_1,g\right]+\left[f_2,g\right],\\
&\left[f_1f_2,g\right]=f_1\left[f_2,g\right]+\left[f_1,g\right]f_2,\\
&\frac{\partial}{\partial t}\left[f,g\right]=\left[\frac{\partial f}{\partial t},g\right]+\left[f,\frac{\partial g}{\partial t}\right].
\end{aligned}$$

The Poisson brackets of the basic variables are easily found to be:

$$\left[q_i,q_k\right]=0,\quad \left[p_i,p_k\right]=0,\quad \left[p_i,q_k\right]=\delta_{ik}.$$

Now, using $[f_1 f_2, g] = f_1[f_2, g] + [f_1, g] f_2$ and the basic variable P.B.'s, $[p, q^2] = 2q$, $[p, q^3] = 3q^2$, and, in fact, the bracket of p with any reasonably smooth function of q is:

$$[p, f(q)] = df / dq.$$

Interlude: a Bit of History of Quantum Mechanics

It should be clear at this point that the Poisson bracket is very closely related to the commutator in quantum mechanics. In the usual quantum mechanical notation, the momentum operator $p = -i\hbar d / dx$, so the commutator (which acts on a wave function, remember)

$$[p, f(x)]\psi = -i\hbar[d / dx, f(x)]\psi = -i\hbar\left(d(f\psi)/dx - fd\psi / dx\right) = -i\hbar\left(df / dx\right)\psi,$$

identical to the Poisson bracket result multiplied by the constant $-i\hbar$.

The first successful mathematical formulation of quantum mechanics, in 1925 (before Schrodinger's equation!) was by Heisenberg. As you know, he was the guy with the Uncertainty Principle: he realized that you couldn't measure momentum and position of anything simultaneously. He represented the states of a quantum system as vectors in some Hilbert space, and the dynamical variables as matrices acting on these vectors. He interpreted the result of a measurement as finding an eigenvalue of a matrix. If two variables couldn't be measured at the same time, the matrices had a nonzero commutator. In particular, for a particle's position and momentum the matrix representations satisfied $[p, x] = -i\hbar$.

Dirac made the connection with Poisson brackets on a long Sunday walk, mulling over Heisenberg's $uv - vu$ (as it was written). He suddenly but dimly remembered what he called "these strange quantities"—the Poisson brackets—which he felt might have properties corresponding to the quantum mathematical formalism Heisenberg was building. But he didn't have access to advanced dynamics books until the college library opened the next morning, so he spent a sleepless night. First thing Monday, he read the relevant bit of Whittaker's *Analytical Dynamics*, and saw he was correct. (From the biography by Helge Kragh.)

Dirac went on to adapt the equation $\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f]$ to quantum mechanics: for time-independent functions, $\frac{df}{dt} = [H, f]$, becomes $i\hbar\dot{f} = [f, H]$ for time development of an operator in the Heisenberg picture, where state vectors of closed systems do not vary in time (as opposed to the Schrodinger picture, where the vectors vary and the operators remain constant).

The Jacobi Identity

Another important identity satisfied by the Poisson brackets is the **Jacobi identity**

$$[f,[g,h]] + [g,[h, f]] + [h,[f, g]] = 0.$$

This can be proved by the incredibly tedious method of just working it out. A more thoughtful proof is presented by Landau, but we're not going through it here. Ironically, the Jacobi identity is a lot easier to prove in its quantum mechanical incarnation (where the bracket just signifies the commutator of two matrix operators, $[a,b]=ab-ba$).

Jacobi's identity plays an important role in general relativity.

Poisson's Theorem

If f,g are two constants of the motion (meaning they both have zero Poisson brackets with the Hamiltonian), then the Poisson bracket $[f,g]$ is *also* a constant of the motion. Of course, it could be trivial, like $[p,q]=1,$ or it could be a function of the original variables. But sometimes it's a new constant of motion. If f,g are time-independent, the proof follows immediately from Jacobi's identity. A proof for time *dependent* functions is given in Landau -- it's not difficult.

Example: Angular Momentum Components

A moving particle has angular momentum about the origin $\vec{L}=\vec{r}\times\vec{p},$ so

$$L_1=r_2p_3-r_3p_2,\quad L_2=r_3p_1-r_1p_3.$$

Using the Poisson brackets found above,

$$\left[r_i,r_j\right]=\left[p_i,p_j\right]=0,\quad \left[p_i,r_j\right]=\delta_{ij},$$

we have

$$\begin{aligned}\left[L_1,L_2\right]&=\left[r_2p_3-r_3p_2,\,r_3p_1-r_1p_3\right]\\&=\left[r_2p_3,r_3p_1\right]+\left[r_3p_2,r_1p_3\right]\\&=-r_2p_1+p_2r_1\\&=L_3.\end{aligned}$$

We conclude that if two components of angular momentum are conserved, so is the third.