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Physics 2415 Lecture 9: Energy in Capacitors

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Work Done in Charging a Capacitor

Suppose we put a charge Q on a capacitance C , thereby raising its potential to $V = Q/C$. Obviously this takes work: as soon as there is any charge on the capacitor, it will repel further charge we put on, so we need to work against that electrostatic repulsion.

To be precise, when the capacitor has charge q it is at potential q/C , and bringing in from far away an incremental additional charge dq requires work equal to the potential energy that small extra charge has gained, that is, $dW = qdq/C$.

The total energy stored in the capacitance once it has charge Q is equal to the total work needed to get the charge there, that is,

$$U = \int_0^Q \frac{q dq}{C} = \frac{Q^2}{2C}.$$

Using $V = Q/C$, $U = \frac{1}{2} Q^2/C = \frac{1}{2} CV^2 = \frac{1}{2} QV$.

This Energy is Stored in the Electric Field!

To show this claim makes sense, we'll consider a few examples, starting with the parallel plate capacitor. Suppose as usual we have plates of area A separated by a distance d which is much smaller than the linear dimensions of the plates, so we have a constant electric field inside (meaning between the plates), and a negligible field from the charged plates outside. Then the energy

$$U = \frac{1}{2} QV = \frac{1}{2} A\sigma Ed = \frac{1}{2} \epsilon_0 E^2 Ad = \frac{1}{2} \epsilon_0 E^2 \times \text{volume}$$

where we used $V = Ed$, $\sigma = \epsilon_0 E$.

This turns out to be always true: an electrostatic field is a store of energy, with energy density $\frac{1}{2} \epsilon_0 E^2$ per unit volume.

*Field Energy for a Charged Sphere

To see that the idea of storing energy in an electric field is not just about parallel plates, consider a spherical conductor of radius R carrying a charge Q . The sphere is then at potential $V = Q/4\pi\epsilon_0 R$, so the energy stored is $U = \frac{1}{2} QV = Q^2 / 8\pi\epsilon_0 R$.

We can derive this value as the total electric field energy density,

$$U = \int \frac{1}{2} \epsilon_0 E^2 dv = \frac{1}{2} \epsilon_0 \int_R^\infty \left(\frac{Q}{4\pi\epsilon_0 r^2} \right)^2 4\pi r^2 dr = \frac{Q^2}{8\pi\epsilon_0 R}.$$

(Here dv is the volume increment.)

*Energy Stored in the Field for Two Charged Spheres

If we have two such spheres, one positive and one negative, far apart, the total energy in the electric field is just twice that for a single sphere. If they're on top of each other, there is no field at all. (one could be a thin skin on the other.) If we draw them distance d apart, $d \gg R$, their own fields dominate out to a distance of d or so, beyond which they cancel, giving the much weaker dipole field. Compared with the potential energy at infinity, then, it's as if they have both lost the field energy corresponding to charged spheres of radius d or so, and since there are two of them, the energy lost in the field is $Q^2 / 4\pi\epsilon_0 d$. This, then, is the energy that must be supplied to get them from d to infinity. Of course, this is a very hand waving argument, and not to be trusted within factors of 2, etc.—but the answer does happen to be right.

Pulling a Disconnected Charged Capacitor Apart

Thinking in terms of energy stored in the electric field gives some insight into the force needed to pull capacitor plates apart. Suppose we pull the plates from separation d to $2d$. Assume first that the capacitor is charged but disconnected, so the charge Q stays the same. Then the field strength

$$E = \sigma / \epsilon_0 = Q / A\epsilon_0$$

remains constant, but there is now an extra volume Ad of field— new field—the field energy storage has doubled.

Question: The increase in energy is $\frac{1}{2} QV = \frac{1}{2} QEd$. But, assuming we held one plate fixed and moved the other, which has charge Q , through distance d , and the electric field between the plates is E , where does the $\frac{1}{2}$ come from?

Answer: The field between the plates, remember, is a superposition of equal fields from the two plates. When we move one plate, it doesn't do work against its own field, which moves with it.

Suppose now both plates were equally **positively** charged: work can be extracted from this system by the plates pushing each other apart. What's going on with the fields now? In this case, there is constant field *outside* the plates, no field in between. As they move apart, the zero field region expands at the expense of the field region, so field disappears. That's where the work comes from.

Pulling a Connected Charged Capacitor Apart

Suppose now we pull apart the plates of a charged capacitor keeping it connected to the battery, so V doesn't change. How does *this* affect the electric field? Since $V = Ed$, if we go from d to $2d$, the field strength drops by a factor of 2, since V is constant, but the **volume** of field doubles. Since the field energy is proportional to E^2 , not E , the total energy stored in the field volume is down by a factor of 2.

This should be no surprise: remember we can write $U = \frac{1}{2}CV^2$, and doubling d halves the capacitance, so at constant V it halves the energy stored.

But it *did* take work to pull the plates apart: where did that energy go? The answer is: into the battery! On halving the capacitance at constant voltage, we lost half the original charge Q . This $\frac{1}{2}Q$ goes into the battery against the voltage V , so the battery is recharged with restored energy $\frac{1}{2}QV$. Only half that energy pumped into the battery comes from energy stores in the capacitor's electric field: the rest comes from work done dragging the plates apart. Let's check that: if the plates have separation x , the field strength $E = V/x$, the field from a single plate is $V/2x$, and the charge on the plates is proportional to E . Therefore, since the initial force on one plate was $QE/2 = Q^2/2\epsilon_0A$. This was at distance d , so at an increased distance x it is down to $(Q^2/2\epsilon_0A)(d^2/x^2)$, and the work needed to double the plate separation is

$$\Delta U = \frac{Q^2}{2\epsilon_0A} \int_d^{2d} \frac{d^2}{x^2} dx = \frac{Q^2d}{4\epsilon_0A} = \frac{Q^2}{4C}.$$

The bottom line is: the work done pulling the plates apart, plus the energy lost thereby from the capacitor, both go into recharging the battery—no energy has disappeared.

Dielectrics

Typically, capacitors have some material between the plates, if only to keep them apart and prevent electrical shorting. This way the plates can be placed closer together, decreasing d and thereby increasing C . But it turns out that the right material increases C in another way as well: all materials respond to some extent to an electric field. Remember the water molecule is an electrical dipole, so in an electric field it will tend to line up with its positive end closer to the negatively charged plate, in a parallel plate capacitor. But even molecules that are not dipoles will become dipoles to some degree in a field, since all molecules have positive nuclei and negative electrons, and the different charges will be displaced in opposite directions by the field. If we think of a neutral solid as a block of positive charge on top of a block of negative charge, this has the effect of moving the blocks in opposite directions, with the result that charge cancellation still holds in the bulk, but there are layers of negative charge and positive charge respectively on the sides next to the positive and negative plates. These charges are of course still bound inside their molecules, so cannot get away—but they have the effect of partially cancelling the charges on the plates Q , so the electric field is lessened, therefore the voltage corresponding to given charge on the plates is less, so the capacitance is enhanced.

The dielectric constant is the factor by which the capacitance is increased on introducing the material to fill the gap between the plates: $C = KC_0$.

As you might guess, it's very high for water (80), but water is not a great choice for dielectric, for obvious reasons. For many ordinary materials (paper, oil, glass) K is around 4, well worth using.

Note that the charges in the dielectric move towards the opposite charges on the capacitor plates, so pulling the dielectric out will take work against this attraction. For a charged but disconnected capacitor, this work goes into building up the larger electric field between the plates corresponding to the lower capacitance but same Q when the dielectric is gone. If the dielectric is removed with the capacitor connected to a battery, the work goes into charging the battery.