home

Physics 2415 Lecture 6: Electrostatic Potential

Michael Fowler, UVa, 9/16/09

The Gravitational Analogy

As we've discussed, the gravitational field from a point mass and the electrostatic field from a point charge both go down with distance as $1/r^2$, and both kinds of fields satisfy the Superposition Principle. It might seem at first glance that electric fields are just going to follow patterns set by gravitational fields, but, of course, there's one huge difference: the two kinds of electric charge allow for both attraction and repulsion, there is no gravitational repulsion between masses. You might think antimatter would repel matter, but that is not the case—all kinds of matter attract gravitationally. (You may have read about Dark Energy in the universe, which apparently causes everything to fly apart. But don't confuse that with Dark Matter, an as yet unidentified kind of matter we know must be there because its gravitational *attraction* is clear from the spinning rate of matter in galaxies.)

Gravitational Potential *mgh* and Its Electrical Equivalent

Let's begin by reviewing the Earth's gravitational field in this room. We can take it to be uniformly downward: a mass *m* will feel a downward force *mg*, doubling the mass doubles the force. That is, the gravitational force on a mass m $\vec{F} = m\vec{g}$ where $\vec{g}$, a downward pointing vector of length *g*, is the gravitational field strength. It takes work to lift a mass m from a point *A* to a higher point *B* against this gravitational pull: to be precise, as discussed earlier, it takes work $W = \int_A^B \left(-m\vec{g}\right)\cdot\overrightarrow{ds}$, where $\overrightarrow{ds}$ is an incremental step on the path, and to move the mass at a steady rate we need to exactly counteract the gravitational force, that is we must exert a force $-m\vec{g}$, so the work done for the step $\overrightarrow{ds} = -m\vec{g}\cdot\overrightarrow{ds}$. Since this is a dot product, the only displacement that requires work is that in the upward direction, and it is easy to see that the total work done against the gravitational force on raising a mass *m* from *A* to *B* is $W = mg\left(h_B - h_A\right)$, where h_B is the height of point *B* above the ground. This work is stored by the system—it can be recovered simply by allowing the mass to fall back, in which scenario it accelerates and thereby gains kinetic energy equal to the work needed to raise it in the first place. The stored energy is called "potential energy".

This leads naturally to the definition of a **gravitational potential** $U\left(h\right) = gh$, so $mU\left(\vec{r}\right) = mgh$ is a measure of the potential energy stored by a mass *m* as a function of position. Following normal usage, we denote height by *h* rather than *z*. There is of course the usual ambiguity concerning what "ground level" we take as $h = 0$, but it is irrelevant in practice as we're always interested in potential energy **differences**.

The electrostatic analogy to gravity near the Earth's surface is the electric field in the region above an infinite, uniformly negatively charged insulating plane.

The electric field has uniform strength and points towards the plane. The force on a charge *q* is $\vec{F} = q\vec{E}$.

Since there is no reason for this plane to be horizontal, we'll measure distance away from the plane as *z*, so $\hat{z}$ is a unit vector normal to the plane. By precise analogy with the gravitational discussion, the work needed to move a charge *q* along a path in this field is $W = \int_A^B \left(-q\vec{E}\right)\cdot\overrightarrow{ds}$, and, without further ado, we can define an electrostatic potential

$$V(z) = Ez = \left(\sigma / 2\varepsilon_0\right) z,$$

where σ is the charge density (this is for an **insulating** charged plane: remember that for a uniform layer of charge σ on the surface of a thick **conductor**, there will be **no factor two** in the denominator).

Now this was a negatively charged plane, so a positively charged particle projected upwards from this plane will follow a parabolic path and come back down, just as a mass will in the gravitational field in this room.

A positively charged particle, on the other hand, will follow a parabolic path upwards! To see this, consider a particle projected parallel to the plane but some distance above it. Particles with the same mass but opposite charges will follow paths that are up-down mirror images of each other.

In practice, a uniform electric field as described above is well approximated in the space between two oppositely charged parallel planes. It is also a good description to the field near the surface of a charged conductor—near enough for the conductor to appear flat.

The Gravitational Analogy at Larger Distances

At distances comparable to the size of the Earth, the gravitational field has the familiar inverse square form $\vec{g}\left(\vec{r}\right) = -GM_E\hat{r}/r^2$.

The work done, and therefore the potential energy difference, on a path in this field is as before

$$W = \int_A^B \left(-m\vec{g}\left(\vec{r}\right)\right)\cdot\overrightarrow{ds}$$

except that the gravitational field is no longer constant. As before, the dot product denotes that **work is only done when there is displacement in the direction of the force**: here **this means displacement in the radial direction**, directly outwards. So, if *A* is at distance r_A from the center of the Earth, and *B* at r_B, the gravitational potential energy difference for a mass *m* = 1 is

$$U\left(\vec{r}_B\right) - U\left(\vec{r}_A\right) = GM_E\int_{\vec{r}_A}^{\vec{r}_B}\frac{\hat{r}\cdot\overrightarrow{ds}}{r^2} = GM_E\int_{r_A}^{r_B}\frac{dr}{r^2} = GM_E\left(\frac{1}{r_B} - \frac{1}{r_A}\right).$$

The very reasonable convention is to take the zero of gravitational potential energy to be at infinity, because in calculating total potential energies, we don't want to have to take account of stars in the next galaxy. This means that, outside the Earth's surface, the gravitational potential energy from the Earth's field is

$$U(\vec{r}) = -\frac{GM_E}{r}, \; r > r_E.$$

A mass resting at the Earth's surface has therefore a negative total (potential plus kinetic) energy, a mass at rest far away has essentially zero total energy—so to get a mass away from the Earth it must be given a kinetic energy sufficient to get it up this potential hill, this corresponds to the escape velocity.

Point Charges and Superposition

Since the electric field is identical in form to the gravitational field, the derivation of gravitational potential energy reviewed above is easily translated into electrostatic potential energy, always bearing in mind that positive and negative charges will have potential energies of opposite sign!

The electrostatic potential difference between two points in the field of a point charge *Q* (or outside a spherically symmetric charge distribution having total charge *Q*) is:

$$V_{\vec{r}_B} - V_{\vec{r}_A} = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{E} \cdot \overrightarrow{ds} = -\frac{Q}{4\pi\varepsilon_0} \int_{\vec{r}_A}^{\vec{r}_B} \frac{\hat{r} \cdot \overrightarrow{ds}}{r^2} = -\frac{Q}{4\pi\varepsilon_0} \int_{r_A}^{r_B} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right).$$

That is, the potential difference is the work done against the electric field per unit charge on moving a charge from point *A* to point *B*.

For a point charge, it is clear that for small enough r the formula must break down (there cannot be infinite energies) but even for electrons within atoms it's very accurate. (It does break down at electron scattering energies reached in particle accelerators, but the process is well understood—the electromagnetic interaction between the point electrons becomes more complicated.)

Thus the potential in the electric field of a point charge is (taking it zero at infinity):

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r}$$

Notice the sign!

If a positive charge is released in the field of a fixed positive charge, it will shoot away, and have kinetic energy far away. This is the **opposite** of the "escape velocity" scenario—that applies for a negative charge in the field of a fixed positive charge.

The **Principle of Superposition works for potential energies** just as it does for electric fields, since the potential energy difference is the sum of contributions from the different fields in the integral, so

$$V\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{r_1}+\frac{Q_2}{r_2}+\frac{Q_3}{r_3}+\cdots\right)$$

and for continuous charge distributions, the sum becomes an integral.

Getting the Electric Field from the Potential

It's often easier to compute the potential than find the electric field for a given charge distribution, since, for the field, one must sum over vectors. So, suppose we have the potential as a function of position. How do we use it to get the electric field at a particular point (x, y, z)?

Write the formula for potential difference between two points separated by an infinitesimal distance dx:

$$V\left(x+dx,y,z\right)-V\left(x,y,z\right)=-\int_{(x,y,z)}^{(x+dx,y,z)}\vec{E}\cdot\overrightarrow{ds}=-E_x dx$$

from which

$$E_x=-\frac{\partial V\left(x,y,z\right)}{\partial x}$$

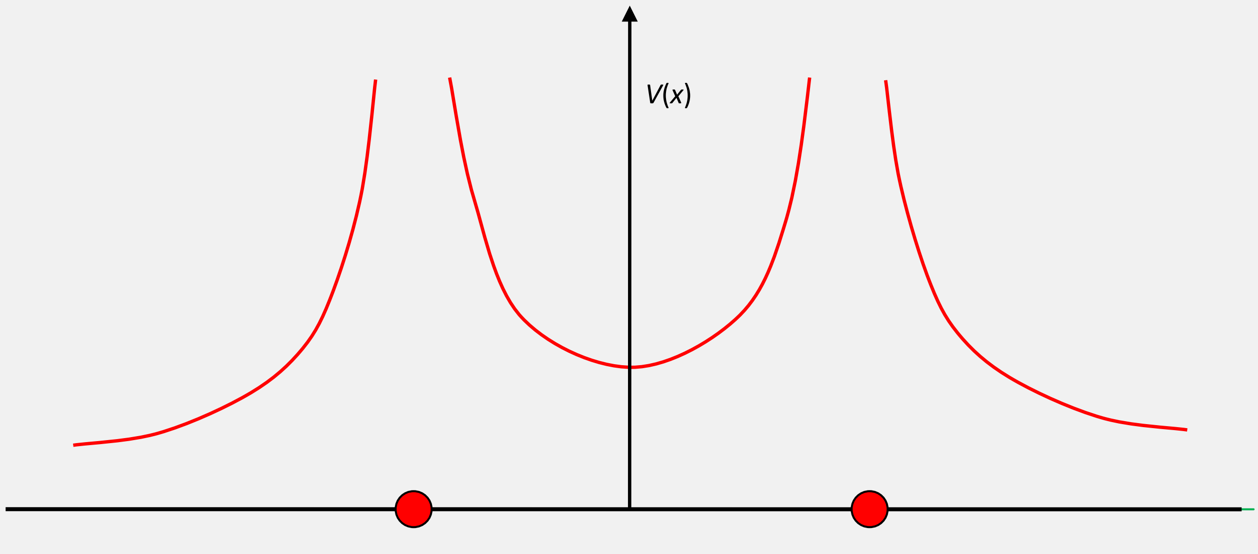
where the special derivative symbol means **partial** differentiation: V is a function of three variables, but we're holding two of them constant—only allowing x to vary.

This formula, plus those in the other two directions, are often combined in a vector notation, written:

$$\vec{E}=-\vec{\nabla}V,\ \text{or}\ \vec{E}=-\mathbf{grad}V.$$

In other words, the electric field in a particular direction is the negative of the slope of the potential in that direction: it's worth looking at a couple of examples to see this in action.

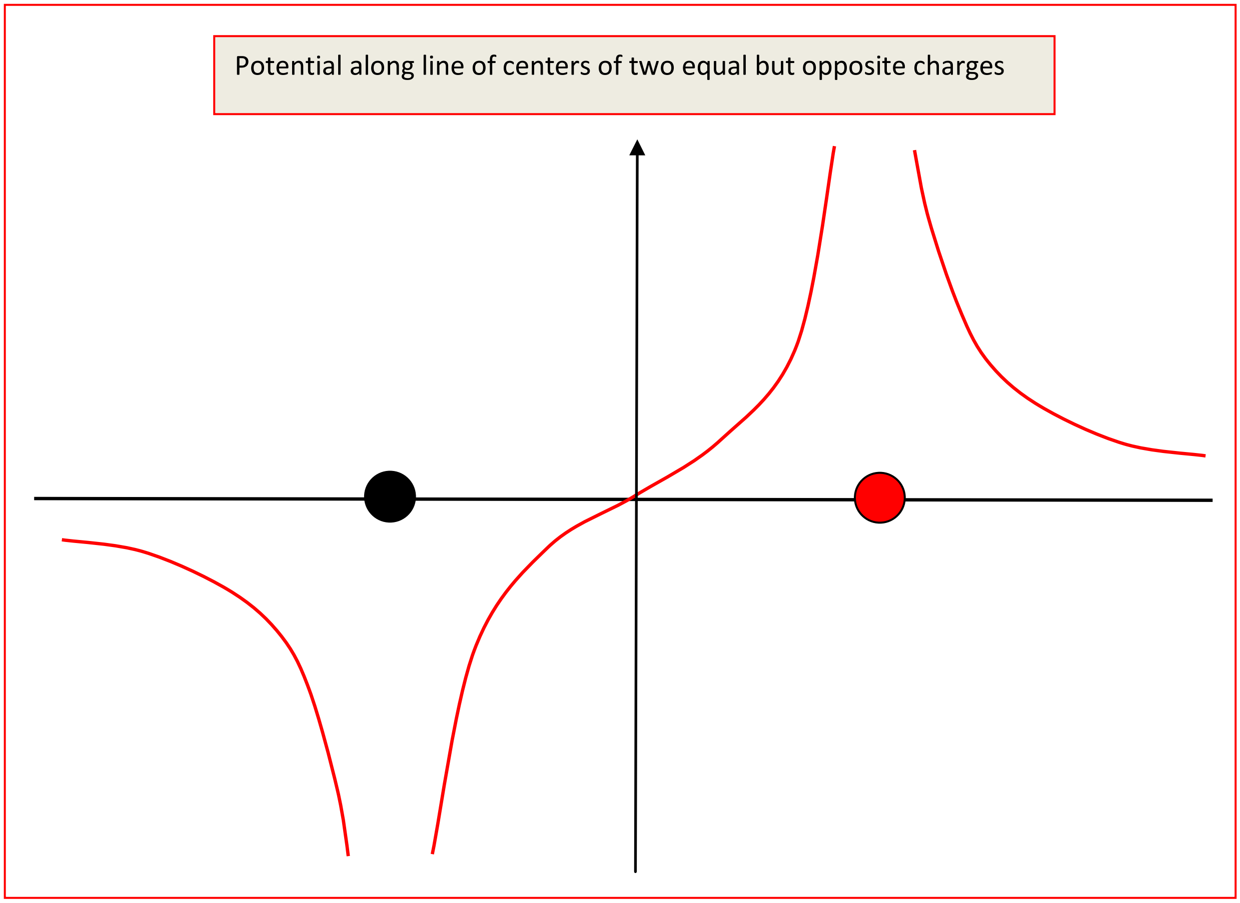
First, consider two equal positive charges, let's say on the x-axis at $+a$ and $-a$, and think about the electric potential and electric field on the axis. This keeps it simple: the electric field points along the axis. The potential plotted along the x-axis looks like:



Potential from two equal positive charges plotted on line through center of charges

Over to the far right, the potential is sloping downwards, so the $\vec{E}$ field is pointing in the positive x-direction. Exactly half way between the charges, the potential bottoms out, that is, its slope is zero: so the electric field is zero at that point—not surprising, since a small positive charge there will be repelled equally by the two positive charges. In fact, the electric field changes sign (along with the potential slope) on going through that point. Note as well, though, that the value of the potential at that low point is not zero: if we moved away from that point in the *y*-direction, it would still be downhill. (Check that by finding the electric field direction at a point on the *y*-axis.)

The second example is a positive charge at $+a$, a negative charge at $-a$ on the *x*-axis. Now the potential along the axis looks like this:



In this case, the electric field between the two charges is strong and in the negative *x*-direction.