home

Physics 2415 Lecture 5: Using Gauss' Theorem: Spheres, Lines, Planes

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Taking Advantage of Symmetry

In general, integrating a vector flux over a surface is a daunting task, but in certain symmetric cases it's very easy, and can then be used to find electric fields much more easily than by adding contributions from large (or infinite, in the case of a continuous distribution) numbers of separate charges.

Spherical Shell

A good example is finding the electric field from a uniformly charged spherical shell, say charge *Q* and radius *R*. Since the sphere is uniformly charged, it has perfect spherical symmetry, it is not altered by turning the sphere through some angle. Therefore, the electric field must also be spherically symmetric. The **only** spherically symmetric electric field has the field pointing directly outwards (or inwards) from the center at all points.

Let's apply $\int_{\text{closed surface}} \vec{E}\cdot\overrightarrow{dA} = (\text{total charge inside surface})/\varepsilon_0$

to a spherical surface of radius *r* bigger than the sphere of charge, but with the same center.

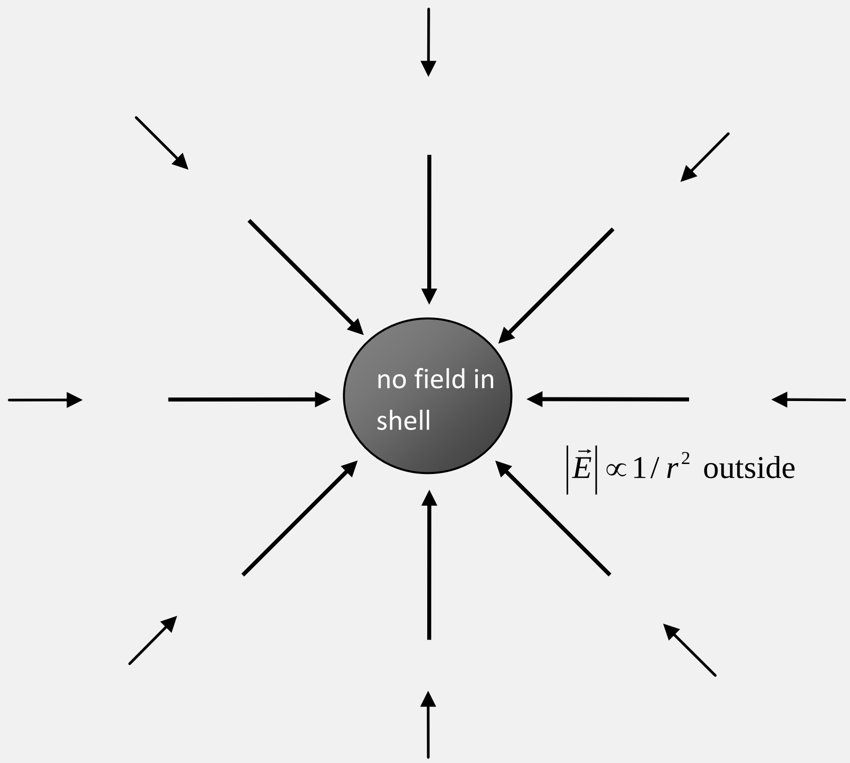
The field $\vec{E}$ points outwards everywhere on the surface, so it's parallel to $\overrightarrow{dA}$, and has the same strength everywhere on the sphere, by symmetry. The total area of the sphere is $4\pi r^2$, so the integral is equal to $4\pi r^2 E$, and outside the sphere of charge:

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

the same as for a point charge at the center. It's worth mentioning that since gravity is also an inverse square force, this same result is true for the gravitational field from a spherical shell of mass. (This can be proved using Coulomb's Law or its gravitational equivalent, but it's quite difficult—it's done here.)

What about the electric field inside the sphere? We do the same trick: integrate over a spherical surface with the same center as the sphere of charge. This time, though, there is no charge inside our smaller spherical surface, so **the electric field must be exactly zero inside the sphere**.

The complete picture of the electric field for a uniformly charged shell is therefore:



Electric field vectors for a uniformly **negatively** charged spherical shell

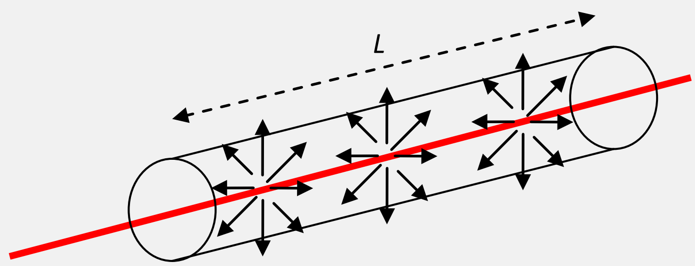
Solid Sphere

The key for any spherically symmetric charge distribution is superposition: the distribution can be expressed as the sum of (or integral over) spherical shells. The contribution from each shell is zero inside that shell, and equal to that from a point charge at the center outside the shell. So, for the case of a uniformly charged (throughout the volume) sphere, outside the whole sphere the field is the same as if all the charge were at the center, inside the solid sphere, at distance r from the center, it's the same field as from a point charge at the center equal to the amount of charge in a sphere of radius r : in other words, there is no contribution from those shells the point is inside. This uniformly charged sphere is not a likely object to find in electrostatics, but it is exactly equivalent to the gravitational field for a sphere of uniform density, a much more realistic problem. And, in fact, the electrostatic uniformly charged sphere was a subject of intense interest a century ago, as a possible model for the atom: before the nucleus was discovered, but it was already known that the atom contained negatively charged electrons, it was suggested that the positive charge was spread over a sphere, and the electrons were inside this sphere: this was called the plum pudding model.

For a nonuniform spherical distribution, the same approach works: the field at any point is equivalent to a point charge at the center equal to all the charge between the point and the center.

Lines and Cylinders of Charge

Gauss' theorem works well for finding the electric field from an infinite uniform line of charge. From symmetry, the field lines must be directed perpendicularly to the line of charge, and the field strength can only depend on distance from the wire. For our Gaussian surface, we take a cylinder of length one meter and radius *r*, the wire running along the axis of the cylinder.



Electric field vectors and Gaussian surface of integration for a positively charged infinite straight wire

The total area of the cylinder is $2\pi r$ so, using

$$\int_{\text{surface}} \vec{E}.\overrightarrow{dA} = (\text{enclosed charge})/\varepsilon_0$$

the enclosed charge,

$$2\pi rLE(r) = \lambda L / \varepsilon_0$$

from which

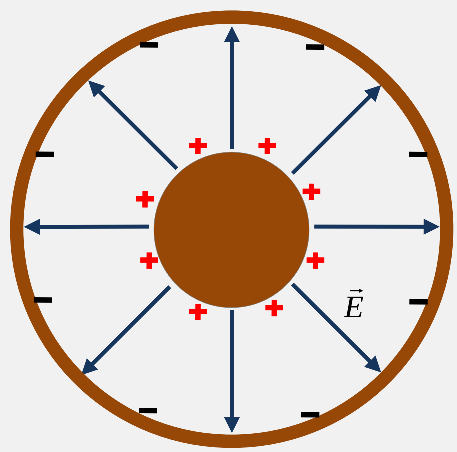
$$E(r) = \frac{\lambda}{2\pi r\varepsilon_0}.$$

(Easier than using Coulomb's Law for the field from each increment of charge and integrating!)

This same method applies for finding the electric field from a uniformly charged cylinder of charge. Just imagine the wire in the picture above being replaced by a fatter wire, then by a hollow cylinder, but staying inside the Gaussian cylindrical surface we integrate over. We get the identical result for E(r), now we must interpret λ as the charge on one meter of the whole cylinder. If this is a hollow cylinder, a pipe, taking a Gaussian surface inside it, the surface encloses no charge, so the electric field inside a hollow cylinder from the charge on the cylinder is zero.

Coaxial Cable

Of course, we could add a line of charge, or even another cylinder, inside our charged cylinder, in which case the total electric field would be the sum of the electric fields from the two cylinders, using superposition. In fact, this is a coaxial cable, the cable used to transmit TV signals. etc. A coaxial cable (the word means "same axis") has a central copper wire, inside a hollow copper cylinder (see figure below). Between the two is a nonconducting dielectric—we'll discuss dielectrics shortly. The transmission of electromagnetic waves, the TV signal, is of course not an electrostatic situation, but nevertheless Gauss' Law still holds, and at any moment there are equal amounts of charge per unit length of cylinder on the surface of the central wire and the inner surface of the cylinder, and consequently an electric field as shown (there are also currents in the copper producing magnetic fields—more about that later).

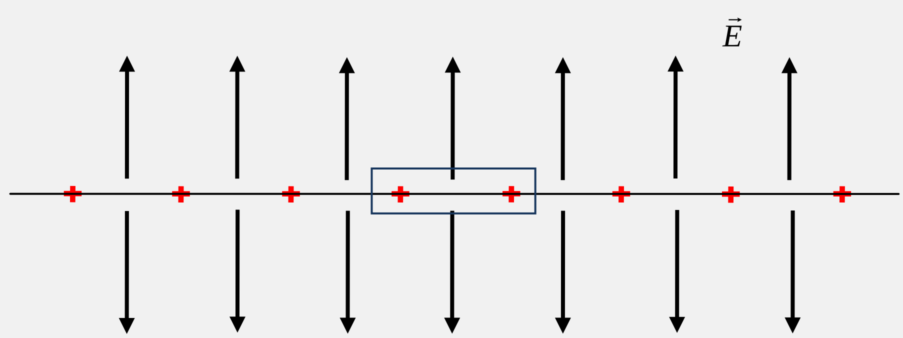


Typical electric field configuration in a coaxial cable, usually a copper cylinder and a central copper wire. The charge is on the **outside** surface of the **inner** conductor, the inside surface of the outer conductor.

The electromagnetic fields are entirely contained inside the cable, in contrast to signals sent down a pair of wires, and the outer cylinder protects the signal from external interference (and makes eavesdropping more difficult—you'll have to cut into the cable).

Uniformly Charged Plane

Gauss' Law makes it extremely easy to find the electric field from a uniformly charged plane, in contrast to the tedious integration necessary using Coulomb's law to find the electric field from each little area of the plane and taking the sum.



Electric field from a uniform plane of charge

From symmetry, taking the plane to have infinite extent, the field must be perpendicular to the plane as shown above, where the plane of charge is seen in cross section, that is, the plane is perpendicular to the paper. Of course, the charge is distributed uniformly over the plane, with area density σ coul/m^2. To use Gauss' Law, we choose a surface shaped like a pillbox, represented in cross section by the rectangle above. The top and bottom surfaces both have area A, and an area A of the charged plane is included. The electric field is parallel to the area vectors on both the top and bottom surfaces, so the total contribution from those surfaces to $\int \vec{E} \cdot \vec{dA} = 2EA$. There is no contribution to the integral from the sides of the pillbox, as the electric field is parallel to those sides, thus $\vec{E} \cdot \vec{dA}$ is zero there. It follows immediately that

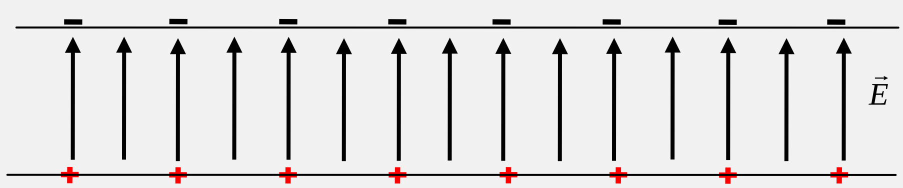
$$\int_{\text{surface}} \vec{E} \cdot \vec{dA} = 2EA = A\sigma / \varepsilon_0, \quad \text{so} \ \ E = \sigma / 2\varepsilon_0.$$

For an actual physical finite plane of charge, this value of *E* is a good approximation at points close to the surface relative to the size of the plane. For distances large compared to the extent of the plane, the field becomes more like that from a point charge.

Parallel Charged Planes; Insulating and Conducting

A much more common scenario is to have **two parallel sheets of charge**, one positive and one negative, having the same charge density.

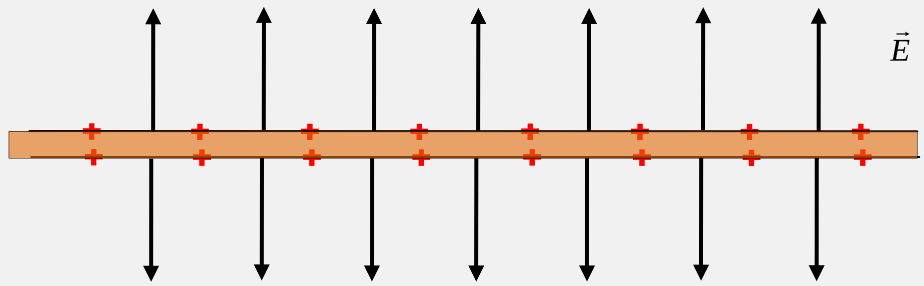
Let us consider first the case where both sheets are **insulators**, the charge has been sprayed on. On bringing the two sheets close, the charges will be unable to move, and the electric fields from the two planes add, from the Principle of Superposition, giving:



Electric field from two uniform parallel planes of charge: one positive, one negative, with equal charge densities σ: the field $E = \sigma / \varepsilon_0$ between the planes, $E = 0$ outside the planes.

Actually, in practice, the sheets are usually **conductors**—in fact, almost all **capacitors** have this basic structure.

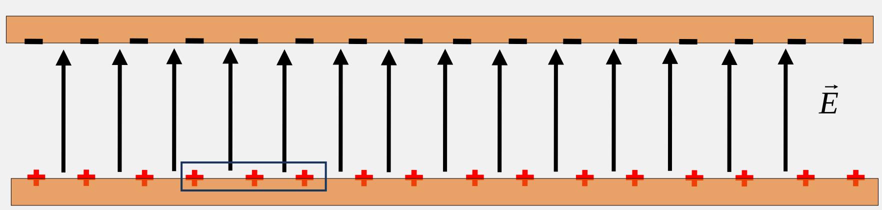
To see how charges move around as conducting charged sheets are brought close, we'll first look at the charge distribution on a single isolated conducting sheet of finite thickness—the charges repel each other, and form equal layers on the two sides:



Electric field from a uniformly charged plane conductor of finite thickness: in the electrostatic situation, there is as always no field inside the conductor, the charges form equal layers on the two sides

Suppose we now take two such conducting planes with equal charge densities, but of opposite signs, and put them close and parallel. What happens?

The positive and negative charges will attract each other, and move to be as close together as possible. That is, all the charges will move to the inside surfaces of the conductors:



Electric field for two uniformly charged plane conductors of finite thickness: no field inside the conductors, or outside the two planes: the charges are moved to the inside surfaces by their mutual attraction.

Note the charge density σ on the lower conductor's top surface generates a field of strength σ / ε_0. This can be understood by considering a pillbox Gaussian surface which encloses that top surface (see diagram): the Gaussian surface has field *E* from its top, but **no electric field** through its bottom, which is **inside** the conductor, where *E* = 0.

Surface Charge on Conductors

The relation between charge density on the surface of a conductor and the electric field just outside the conductor, $E = \sigma / \varepsilon_0$, derived in the previous example, is in fact true in general. That is, any charged conductor in an electrostatic situation has a surface charge density, and the electric field immediately outside the surface is perpendicular to it and has strength σ / ε_0. Even if the surface is not locally flat, a small enough Gaussian pillbox surface can be drawn to prove this. Remember there is no electric field inside the conductor.

The absence of an electric field inside the conductor means there can be no net charge contained in any closed surface lying entirely inside the conducting material. If there are holes in the material inside this surface, and charge is placed in these holes, charges will move inside the conductor to the surfaces of these holes, to exactly compensate for the charges placed in the holes. Lines of force from these added charges will terminate at the surfaces of the holes.