More Angular Momentum, then Statics

Physics 1425 Lecture 23

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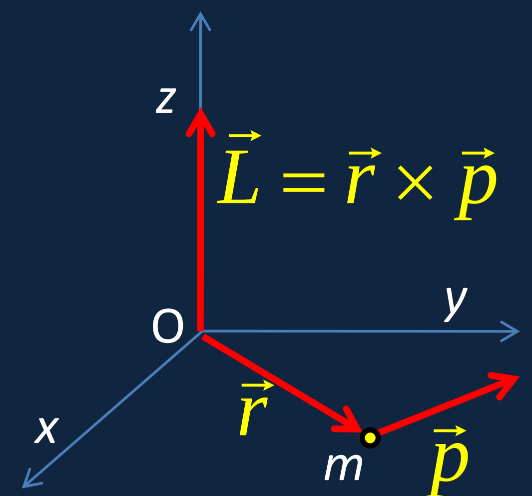
Vector Angular Momentum of a Particle

- A particle with momentum $\vec{p}$ is at position $\vec{r}$ from the origin O.
- Its angular momentum about the origin is

$$\vec{L} = \vec{r} \times \vec{p}$$

- This is in line with our definition for part of a rigid body rotating about an axis: *but also works for a particle flying through space*.



Viewing the *x*-axis as coming out of the slide, this is a "right-handed" set of axes: $\hat{i} \times \hat{j} = +\hat{k}$

Clicker Question

- A particle moves along a straight line at constant speed. The line does not pass through the origin. Is the particle’s angular momentum about the origin constant?

A. Yes

B. No

Clicker Answer

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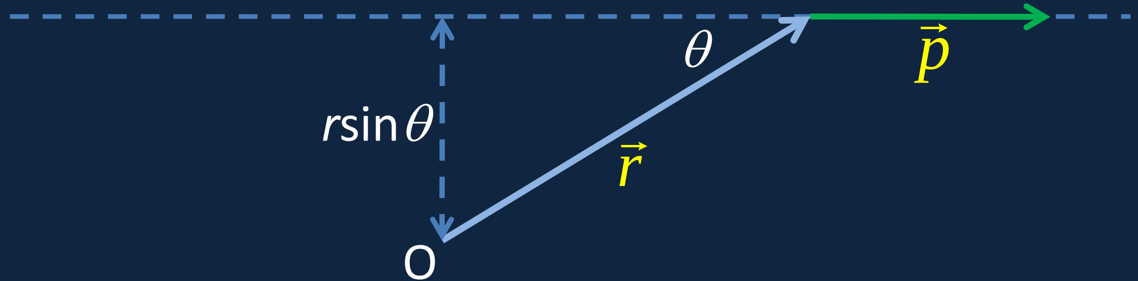
A. <u>Yes:</u> $\left|\vec{L}\right| = \left|\vec{r} \times \vec{p}\right| = \left(r\sin\theta\right)p$ and r$\sin\theta$ is just the perpendicular distance of the line of motion from the origin—this is the same for any point on the line.



Rotational Motion of a Rigid Body

- For a collection of interacting particles, we've seen that

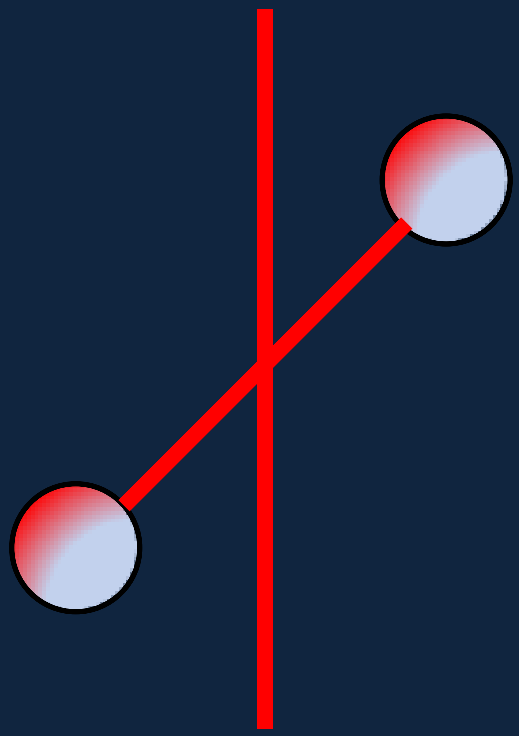
$$d\vec{L} / dt = \sum_i \vec{\tau}_i$$

the vector sum of the applied torques, $\vec{L}$ and the $\vec{\tau}_i$ being measured about a fixed origin O.

- A rigid body is equivalent to a set of connected particles, so the same equation holds.
- It is also true (proof in book) that even if the CM is accelerating,

$$d\vec{L}_{\text{CM}} / dt = \sum \vec{\tau}_{\text{CM}}$$

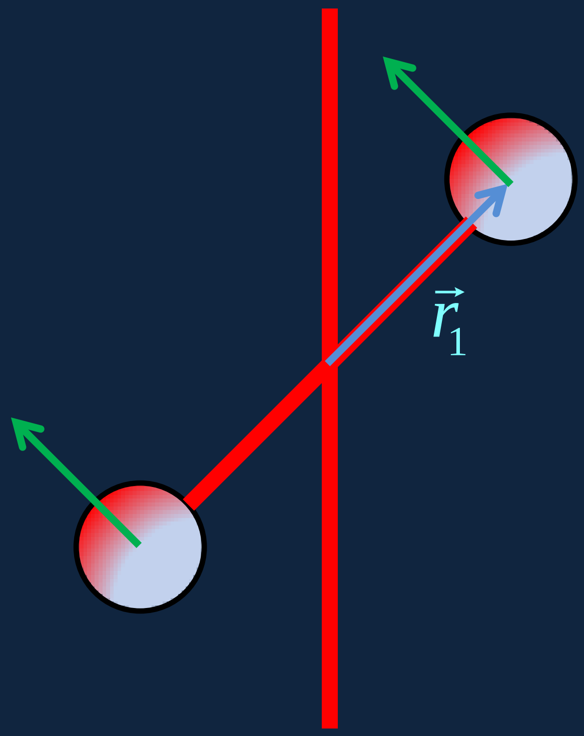
A dumbbell (two small masses at the ends of a light rigid rod) is mounted on a fixed axle through its center, at an angle θ. It is set in steady rotation. The direction of the angular momentum of the system is:



A. Along the axle
B. Along the dumbbell rod
C. Neither of the above.

Clicker Answer

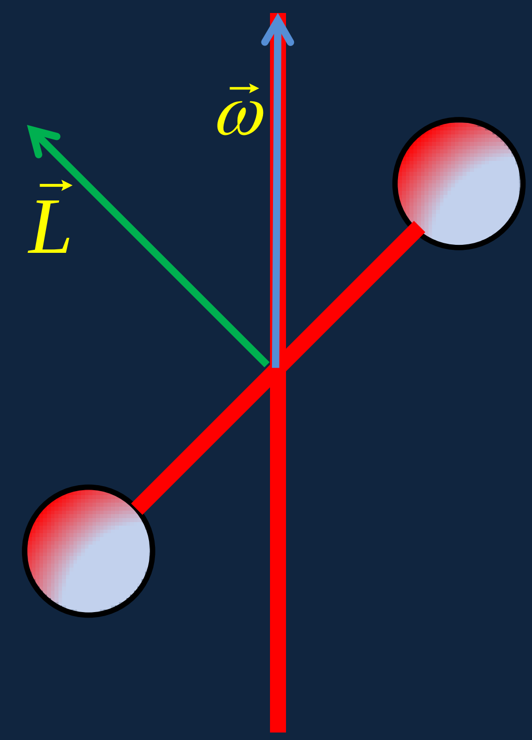
- The angular velocity vector $\vec{\omega}$ is vertical.
- The total angular momentum $\vec{L}$ about the CM is $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$.
- Assume we're looking at the rotation at the instant when the ball to the right is moving directly inwards (into the screen). Then $\vec{r}_1 \times m\vec{v}_1$ is in the plane of the screen, as shown. BUT $\vec{r}_2 \times m\vec{v}_2$ is in the same direction! So $\vec{\omega}$ is constant, but $\vec{L}$ is rotating.
- This means the axle is supplying a rotating torque to the dumbbell. This is what causes problems with unbalanced tires.



Wait a minute—isn't $\vec{L} = I\vec{\omega}$?
See next slide!

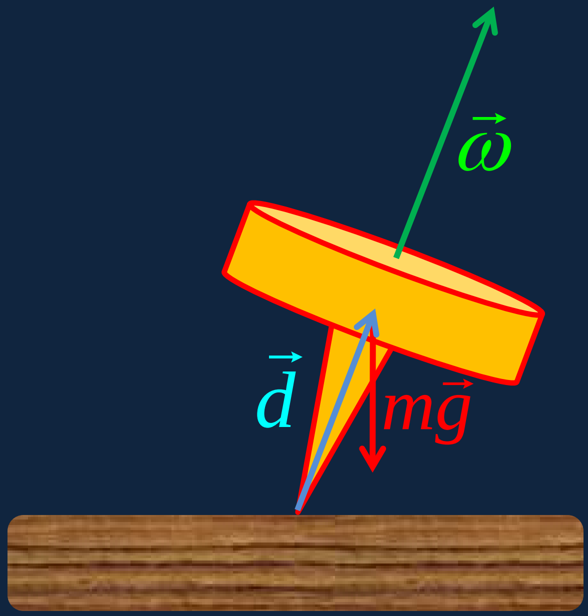
A Bit More About $\vec{L}$ and $\vec{\omega}$...

- We've used $\vec{L} = I\vec{\omega}$ a lot.
- We see from this example it's not always true that $\vec{L}, \vec{\omega}$ are parallel vectors.
- What's going on?
- The answer is that $\vec{L}, \vec{\omega}$ are only parallel if the spinning body is symmetric about the axis of rotation—which is usually the case.
- For more complicated cases, you will still see $\vec{L} = \mathbf{I}\vec{\omega}$, but that fat $\mathbf{I}$ denotes a tensor or matrix.



Spinning Top

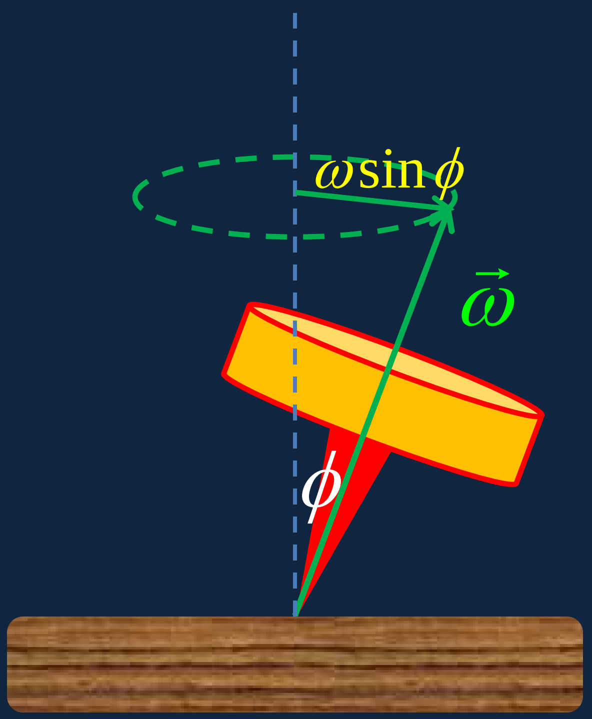
- Pointing your right thumb in the direction of the angular velocity vector $\vec{\omega}$, your curling fingers point in the direction of rotation.
- Gravity exerts a torque about the pivot point $\vec{\tau} = \vec{d} \times m\vec{g}$, evidently directed inwards.
- From $\vec{\tau} = d\vec{L} / dt = Id\vec{\omega} / dt$ $d\vec{\omega}$ will be inwards, the tip of $\vec{\omega}$ is describing a horizontal circle: this is "precession".



Precession Rate

$$\vec{\tau} = d\vec{L}/dt = Id\vec{\omega}/dt$$

- The horizontal component of the angular velocity vector $\vec{\omega}$ has length $\omega \sin\phi$ and it precesses around a circle centered above the pivot point.
- The precession angular velocity is written $\Omega = d\theta/dt$, where θ measures angle around the horizontal circle.
- If in time *dt* there is precession through $d\theta$, $d\omega = (\omega \sin\phi)d\theta$
- so $$\frac{d\theta}{dt} = \frac{1}{\omega \sin\phi}\frac{d\omega}{dt} = \frac{1}{\omega \sin\phi}\frac{\tau}{I} = \frac{mgd}{\omega I}$$



Statics: Conditions for Equilibrium

- For any body, $M d\vec{v}_{\mathrm{CM}} / dt = \sum \vec{F}_i$, the net force causes the CM to accelerate. Hence, if the body is remaining at rest,

$$\sum_i \vec{F}_i = 0$$

- To eliminate *angular* acceleration, there must be zero torque about any axis. If all forces are in one plane, it's enough to prove zero torque about one axis perpendicular to the plane:

$$\sum_i \vec{\tau}_i = 0$$

Free Body Diagrams

- To apply Newton's Laws to find how a body moves, we must focus on that body alone and add all the (vector) forces acting on it.
- The diagram showing all the forces on one body (or even part of a body) is called a "free body diagram"—we've "freed" the body from the rest of the system, representing everything else just by the forces on this body.
- The net (total) force then goes into $\Sigma\vec{F} = m\vec{a}$.

Flat Forces?

- If a body in equilibrium is acted on by three and only three forces, do the force vectors have to lie in a plane?

A. Yes

B. No

Flat Forces

- If just three forces are acting on a body, and it's in equilibrium, they must all lie in the same plane, because if we choose the plane defined by two of them, and the third force has a component perpendicular to that plane, nothing is balancing this perpendicular force.

Clicker Question

- A body is in equilibrium. It is acted on by three forces, lying in a plane.
- Do the lines of action of the three forces all pass through the same point?

A. Yes

B. No

Three Force Equilibrium

- If a body is in equilibrium when acted on by three forces, the three forces must lie in the same plane AND all pass through a common point. If they don't, taking a perpendicular axis through a point where two of them meet, the third force gives an unbalanced torque about that point, so the body will have angular acceleration.

Clicker Question

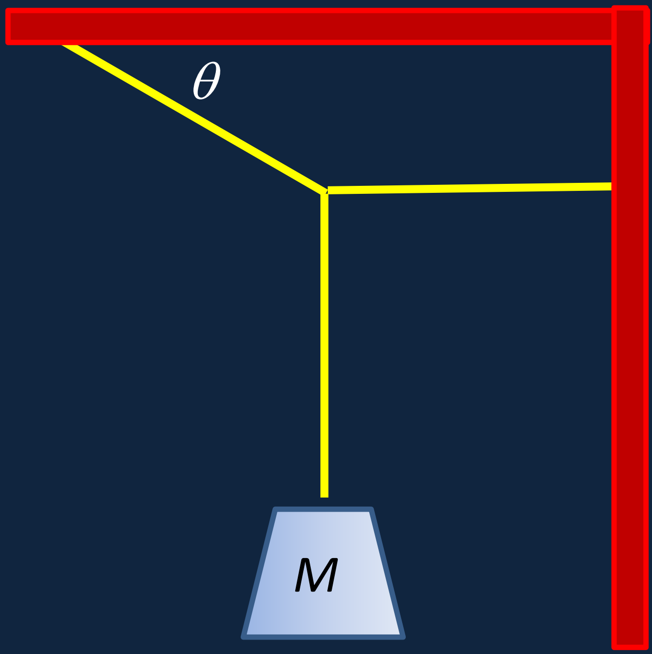
- What is the tension T in the horizontal string?

A. $Mg\cos\theta$

B. $Mg\tan\theta$

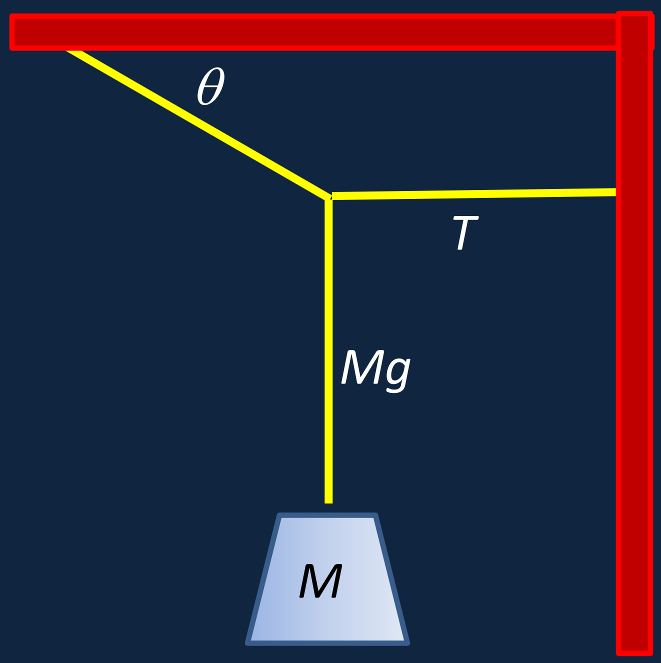
C. $Mg\cot\theta$

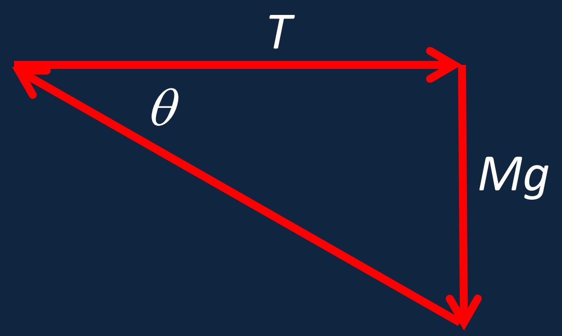
D. None of the above.



Clicker Answer

Free body diagram for the knot where the three strings meet: the vector tension forces on it must add to zero.





Evidently, $Mg/T = \tan\theta$, $T = Mg\cot\theta$.

Clicker Question

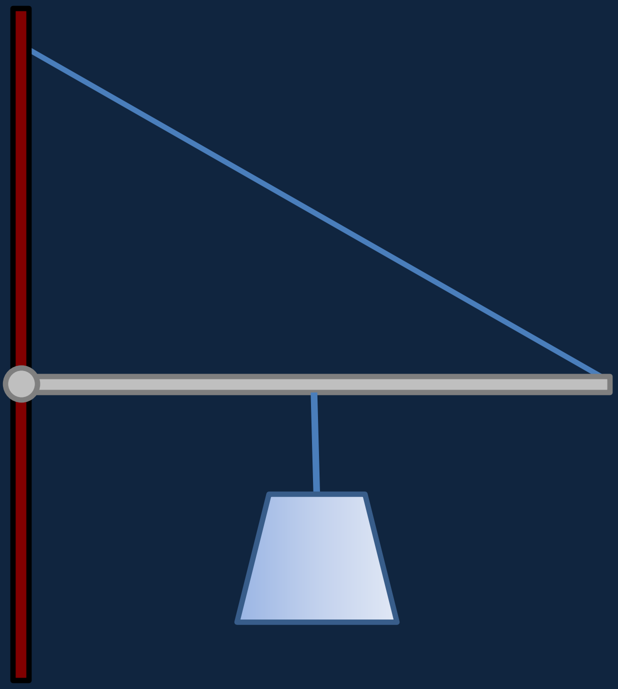
- What is the approx tension *T* in the top string, given the mass is 2 kg, and it's hung from the midpoint of the rod, which is light and hinged, the angle is 30°?

A. 10 N

B. 20 N

C. $20\sqrt{3}$ N

D. 40 N



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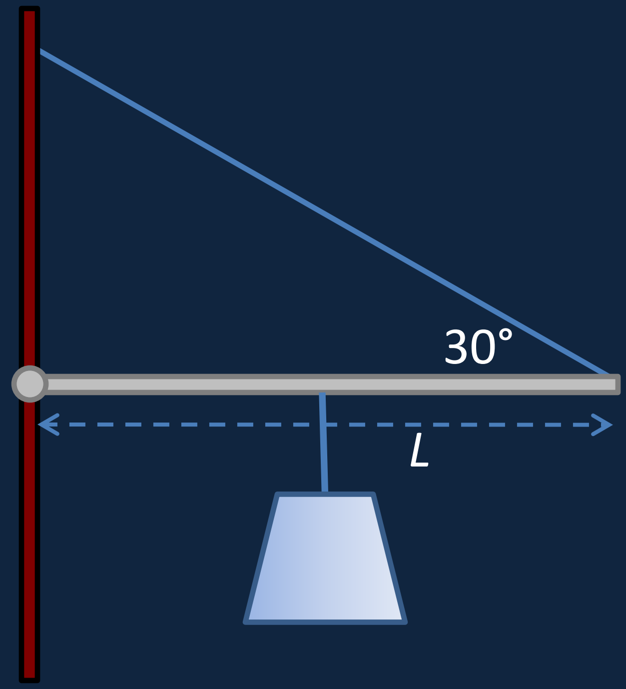

The distance from the hinge to the line of action of the force is $L\sin 30 = L/2$.
Alternatively, the component of the tension force perpendicular to the rod is $T\sin 30 = T/2$.