Rotational Dynamics

Physics 1425 Lecture 19

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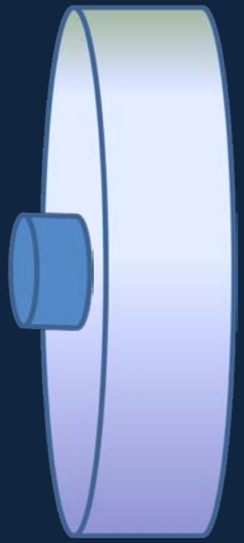
Rotational Dynamics

- Newton's First Law: a rotating body will continue to rotate at constant angular velocity as long as there is no torque acting on it.
- Picture a grindstone on a smooth axle.
- BUT the axle must be *exactly* at the center of gravity—otherwise gravity will provide a torque, and the rotation will not be at constant velocity!



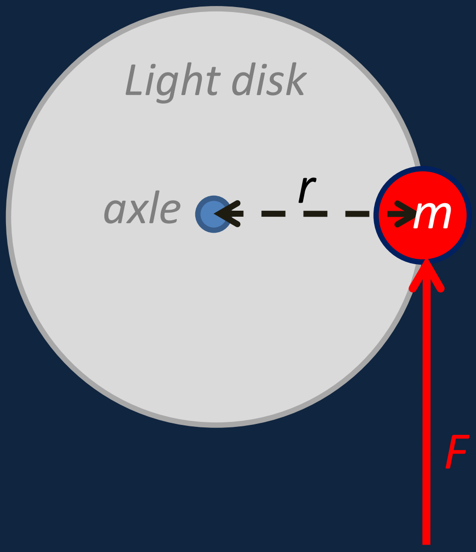
How is Angular Acceleration Related to Torque?

- Think about a tangential force *F* applied to a mass *m* attached to a light disk which can rotate about a fixed axis. (A *radially* directed force has zero torque, does nothing.)
- The relevant equations are:

  $F = ma$, $a = r\alpha$, $\tau = rF$.

- Therefore $F = ma$ becomes

$$\tau = mr^2\alpha$$



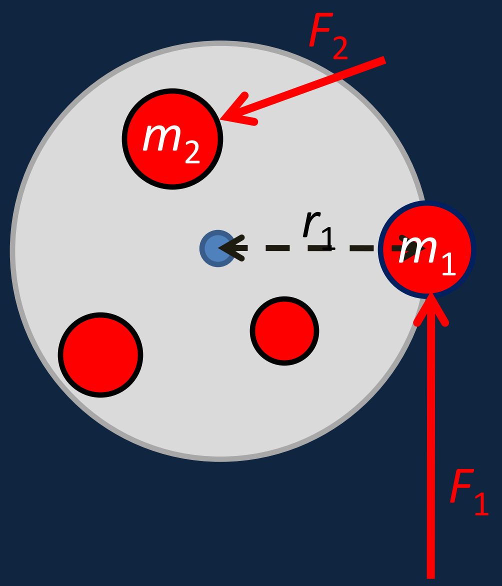
Newton's Second Law for Rotations

- For the special case of a mass m constrained by a light disk to circle around an axle, the angular acceleration α is proportional to the torque τ exactly as in the linear case the acceleration a is proportional to the force F:
- $\tau = mr^2\alpha \longleftrightarrow F = ma$

The angular equivalent of inertial mass m is the moment of inertia mr^2.

More Complicated Rotating Bodies

- Suppose now a light disk has several different masses attached at different places, and various forces act on them. As before, radial components cause no rotation, we have a sum of torques.
- BUT the rigidity of the disk ensures that a force applied to one mass will cause a torque on the others!
- How do we handle that?



Newton's Third Law for a Rigid Rotating Body

- If a rigid body is made up of many masses m_i connected by rigid rods, the force exerted along the rod of m_i on m_j is equal in magnitude, opposite in direction and along the same line as that of m_j on m_i, therefore the internal torques come in equal and opposite pairs, and cannot contribute to the body's angular acceleration.
- It follows that the angular acceleration is generated by the sum of the external torques.

Moment of Inertia of a Solid Body

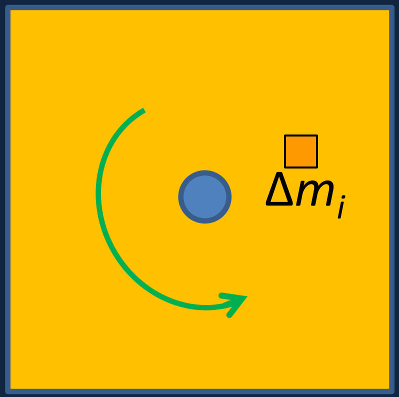
- Consider a flat square plate rotating about a perpendicular axis with angular acceleration α. One small part of it, Δm_i, distance r_i from the axle, has equation of motion

$$\tau_i = \tau_i^{\text{ext}} + \tau_i^{\text{int}} = \Delta m_i r_i^2 \alpha$$

- Adding contributions from all parts of the wheel

$$\tau = \sum_i \tau_i^{\text{ext}} = \left(\sum_i \Delta m_i r_i^2 \right) \alpha = I\alpha$$

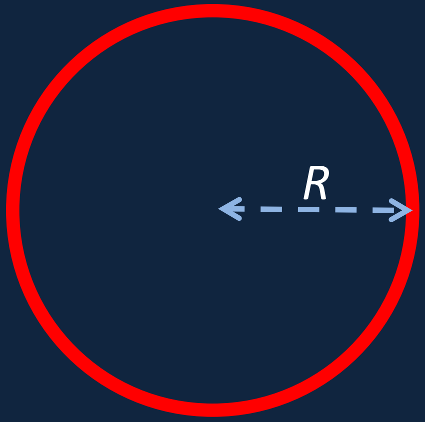
- I is the Moment of Inertia.



Calculating Moments of Inertia

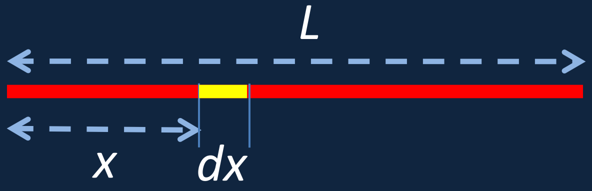
- A thin hoop of radius *R* (think a bicycle wheel) has all the mass distance *R* from a perpendicular axle through its center, so its moment of inertia is

$$I = \sum_i \Delta m_i r_i^2 = MR^2$$



- A uniform rod of mass *M*, length *L*, has moment of inertia about one end

$$I = \int_0^L x^2 (M/L)\,dx = \tfrac{1}{3} ML^2$$



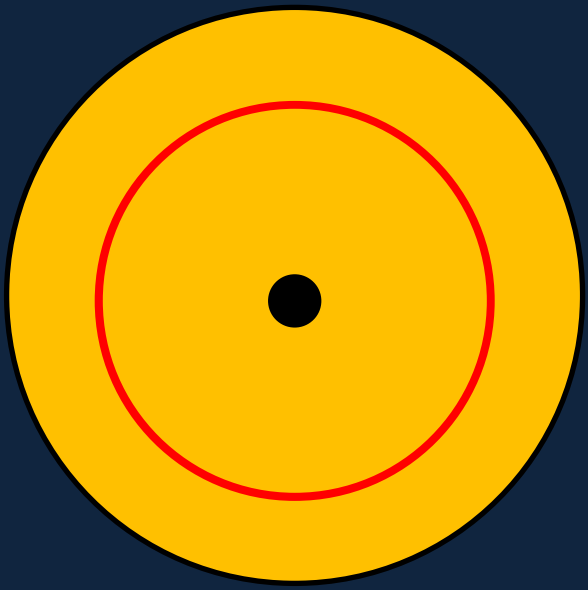
Mass of length *dx* of rod is (*M*/*L*)*dx*

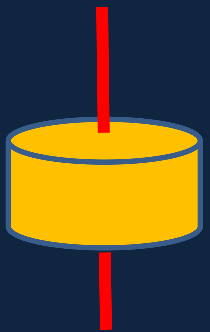
Disks and Cylinders

- A disk: mass *M*, radius *R*, is a sum of nested rings.
- The red ring, radius *r* and thickness *dr*, has area $2\pi r dr$, hence mass $dm = M(2\pi r dr / \pi R^2)$.
- Adding up rings to make a disk,

$$I = \int_0^R r^2 dm = \int_0^R r^2 \left(2M / R^2\right) r dr = \tfrac{1}{2} MR^2$$

- A cylinder is just a stack of disks, so it's also $\tfrac{1}{2}MR^2$ about the axle.





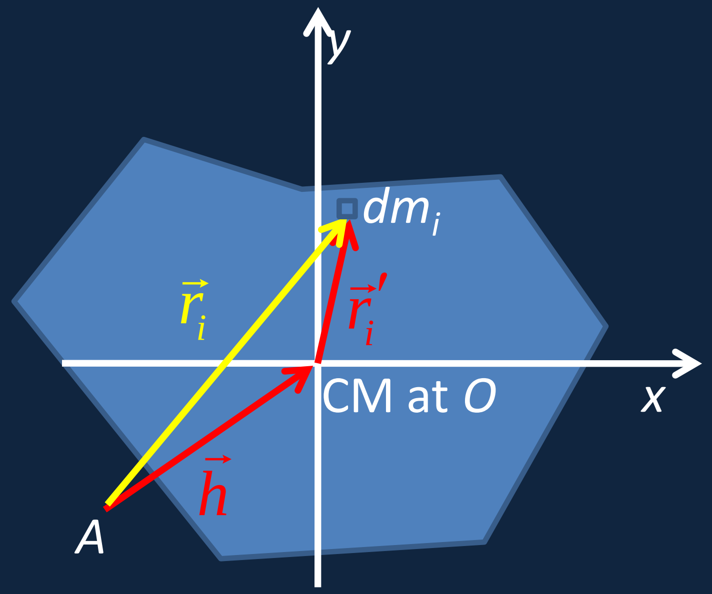
Parallel Axis Theorem

- If we already know I_{CM} about some line through the CM (we take it as the *z*-axis), then *I* about a parallel line at a distance *h* is
- $$I = I_{\text{CM}} + Mh^2$$
- Here's the proof:

$$I = \sum_i m_i \vec{r}_i^{\,2} = \sum_i m_i \left(\vec{r}_i' + \vec{h}\right)^2$$

$$= \sum_i m_i \vec{r}_i^{\,\prime 2} + 2\vec{h} \cdot \sum_i m_i \vec{r}_i' + M\vec{h}^2$$

$$= I_{\text{CM}} + Mh^2 \quad \left(\text{Since } \sum_i m_i \vec{r}_i' = 0.\right)$$



Moment of inertia *I* about perpendicular axis through *A*

- We prove it for a 2D object—the proof in 3D is exactly the same, taking the line through the CM as the *z*-axis.

Clicker Question

We found the moment of inertia of a rod about a perpendicular line through one end was $\frac{1}{3}ML^2$.
Use the parallel axis theorem to figure out what it is about a perpendicular line through the center of the rod.

A $\frac{1}{3}ML^2$

B $\frac{7}{12}ML^2$

C $\frac{1}{2}ML^2$

D $\frac{1}{4}ML^2$

E $\frac{1}{12}ML^2$

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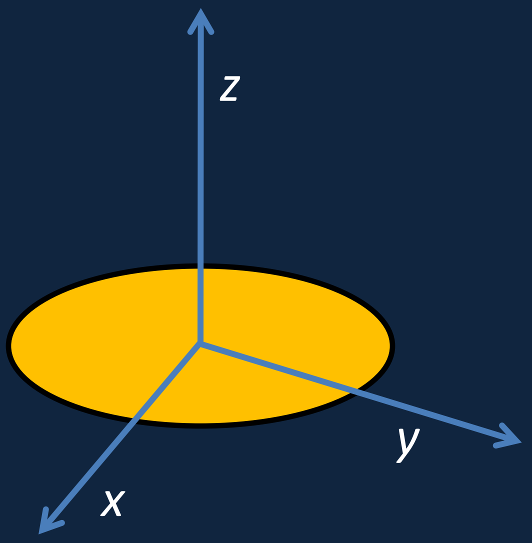
The moment of inertia about the CM is less than about any other parallel axis—the mass is closer to the axle on average.

Perpendicular Axis Theorem

- For a 2D object (a thin plate) the moment of inertia I_z about a perpendicular axis equals the sum of the moments of inertia about any two axes at right angles through the same point in the plane,

$$I_z = I_x + I_y$$



- Proof:

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i \left(x_i^2 + y_i^2 \right) = I_x + I_y$$

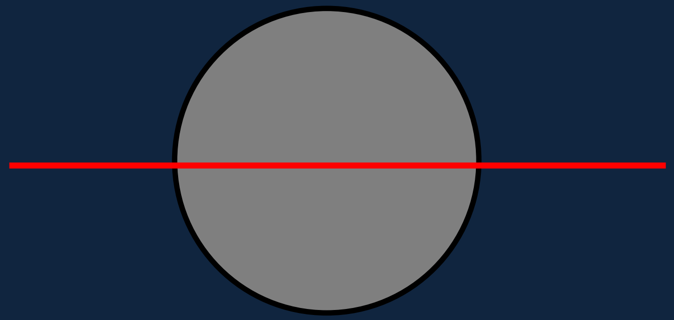
Clicker Question

Given that the moment of inertia of a disk about its axle is $\frac{1}{2}MR^2$, use the perpendicular axis theorem to find the moment of inertia of a disk about a line through its center and in its plane.

A $\frac{1}{2}MR^2$

B $\frac{1}{4}MR^2$

C MR^2



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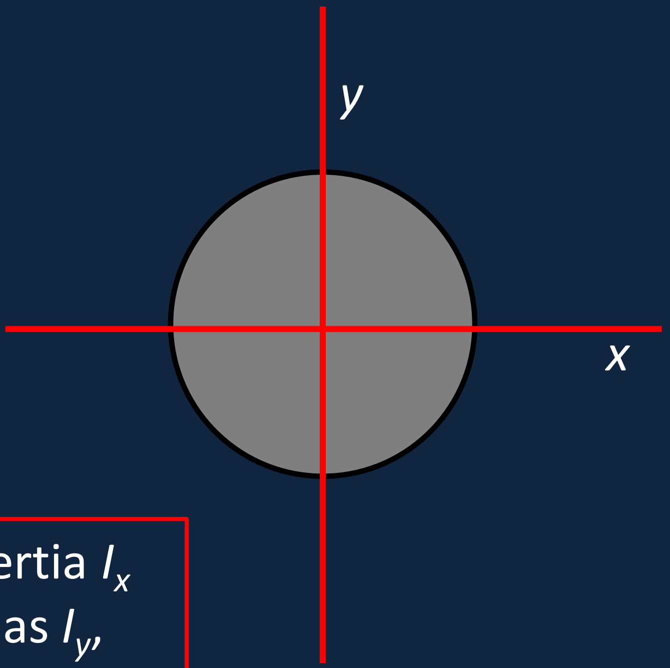

From symmetry, the moment of inertia I_x about the x-axis must be the same as I_y, and from the perpendicular axis theorem, $I_z = I_x + I_y$.

Rotational Kinetic Energy

- Imagine a rotating body as composed of many small masses m_i at distances r_i from the axis of rotation.
- The mass m_i has speed $v = \omega r_i$, so $KE = \frac{1}{2} m_i r_i^2 \omega^2$.
- The total KE of the rotating body (assuming the axis is at rest) is

$$K = \sum_i \left(\tfrac{1}{2} m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I\omega^2$$