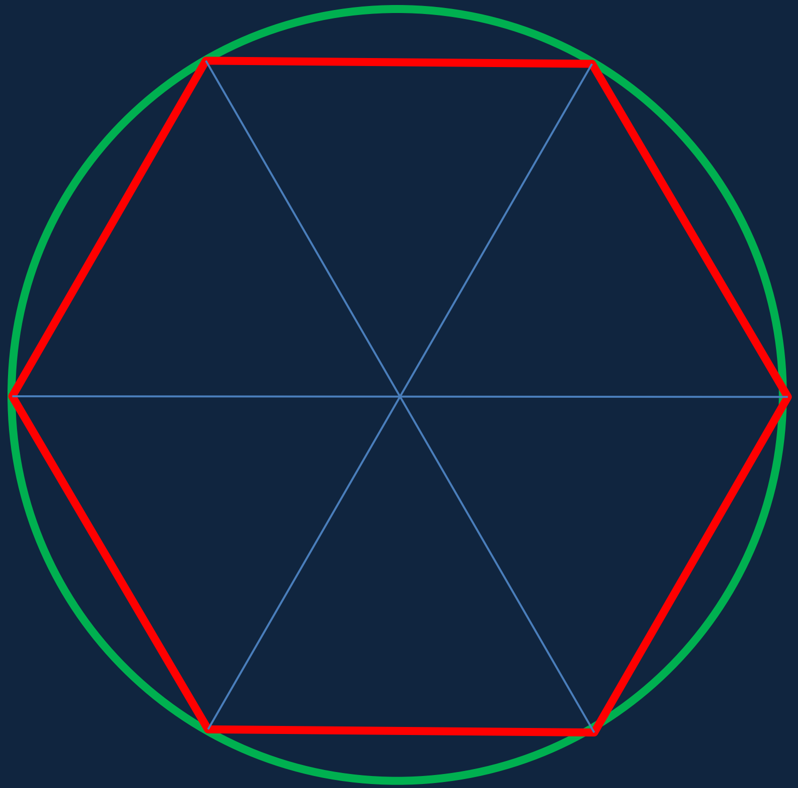
Circular Motion

Physics 1425 Lecture 18

Michael Fowler, UVa

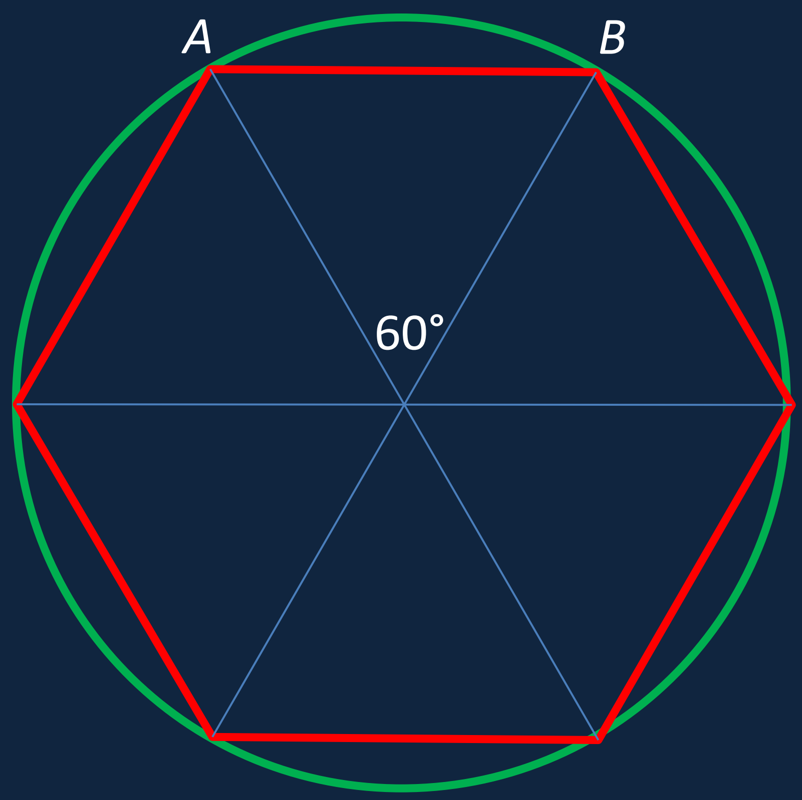
How Far is it Around a Circle?

- A regular hexagon (6 sides) can be made by putting together 6 equilateral triangles (all sides equal).
- The radius of the circle = 1.
- The distance all the way round the hexagon (red path) = 6.
- The distance all the way round the circle (green path) is a little more: in fact, it's **$2\pi r$ = 6.283…**



Arcs Subtending Angles: the Radian

- It's 360° all the way round the circle, that's 60° from each of the equilateral triangles.
- We say that the arc of circle between *A* and *B* "subtends" an angle of 60° at the center of the circle.
- One <u>radian</u> is defined as the <u>angle</u> subtended by an arc equal in length to the radius of the circle.



Clicker Question

One radian is:

A. 60°
B. 120°
C. A bit less that 60°
D. A bit more than 60°
E. None of the above

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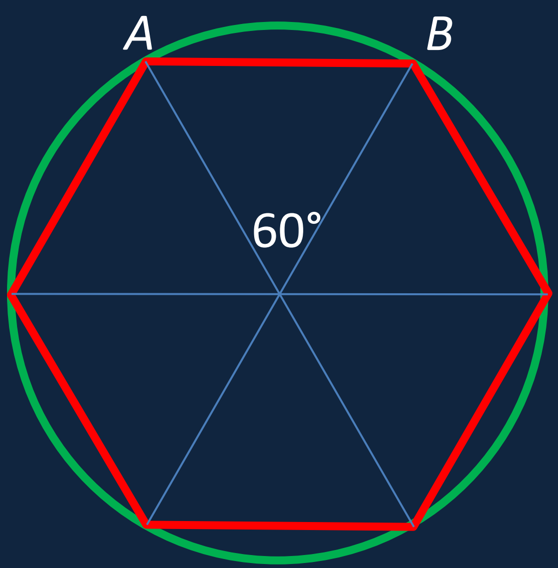


The straight line distance from *A* to *B* is one side of an equilateral triangle, exactly one radius, the arc from *A* to *B* is a bit further—so 60° is a little *more* than one radian.

Full Circle

- For a circular path of radius *r*, if you walk a distance *r* along the path, you have gone around an angle of one radian relative to the center.
- If you walk all the way around the path, you have of course gone through 360°.
- BUT you've walked a total distance $2\pi r$, and therefore around an angle of 2π radians.
- Conclusion: **360° = 2π radians**

Radians and Trig

- Measuring the angle θ in radians,

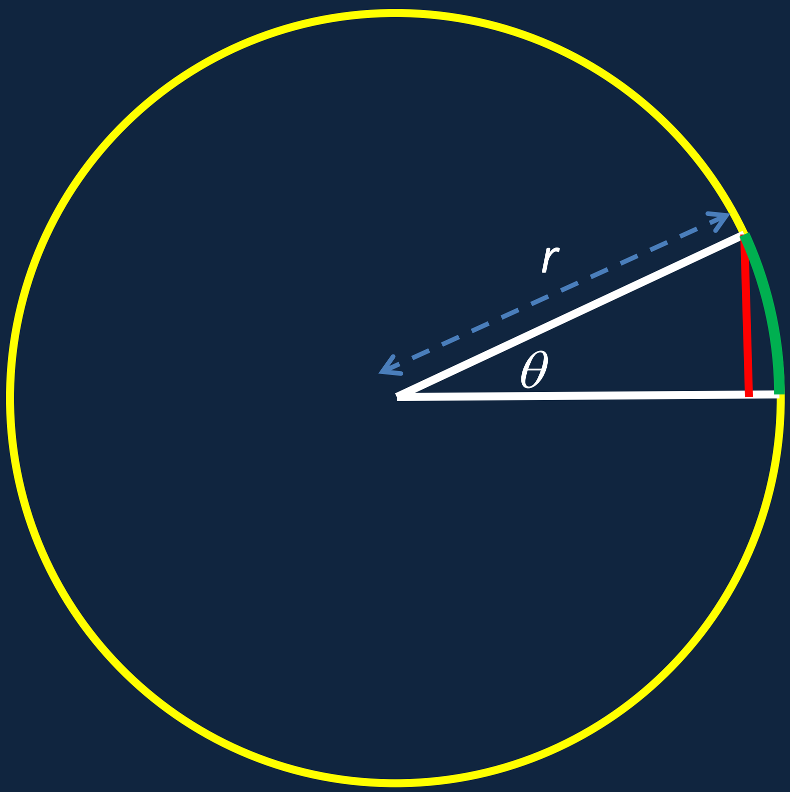
- θ = (length green arc)/r

  and

- $\sin\theta$ = (length red line)/r

- so for small angles

  $\sin\theta \approx \theta$

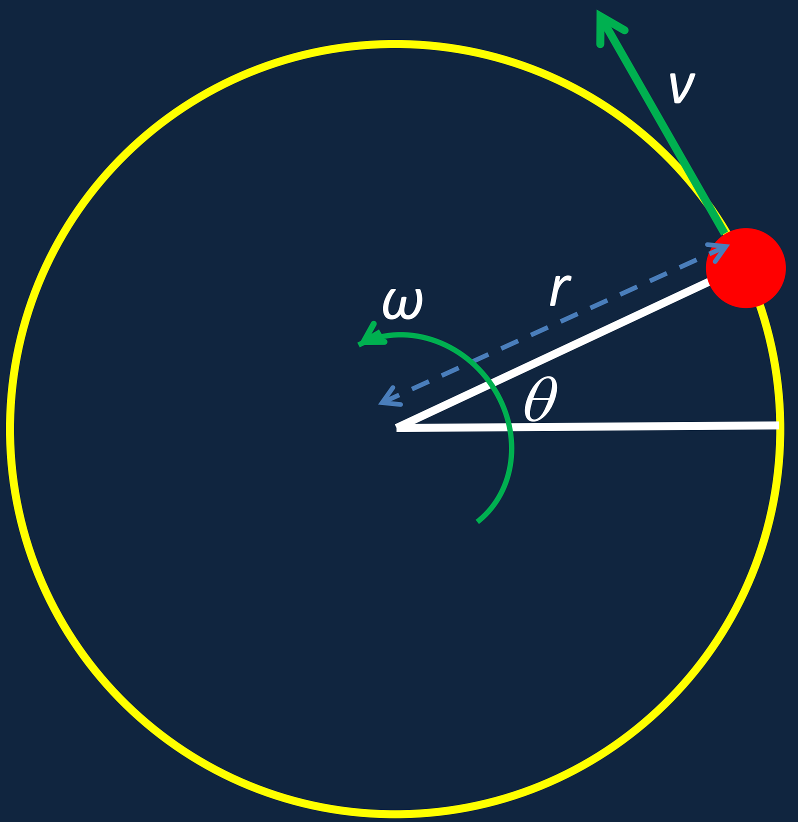


Units for Angular Velocity

- How fast is something rotating?
- Car engine: units rpm, revs per minute, redlines around 6,000 rpm or 100 revs/sec.
- 1 Hertz, written 1Hz, means one cycle/sec, used for electrical generators, circuits. (Often called the rotational frequency, and written *f*.)
- Second hand on watch turns at 1 rpm, or 6°/sec.
- Earth goes round Sun at very close to 1°/day
- (probably why the degree was the original measure of angle.)

Angular Speed and Rim Speed

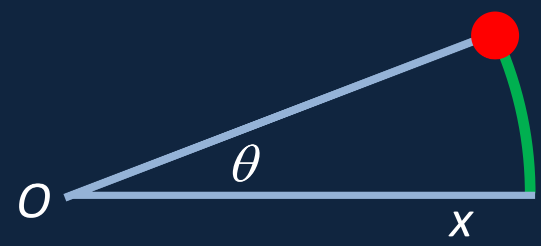
- If a wheel of radius r rotates one revolution per second, a ball on the rim is moving at speed $v = 2\pi r$ m/sec.
- If it rotates at one radian per sec, $v = r$ m/sec.
- If it rotates at ω rad/sec, $v = \omega r$ m/sec.
- we'll measure angular velocities in radians per second and often use
- $$v = \omega r$$



Note: $\omega = 2\pi f$, if f is the frequency in cycles per second.

Standard Angular Notation

- Angle: theta, θ, in radians, measured <u>counterclockwise</u> from the *x*-axis.



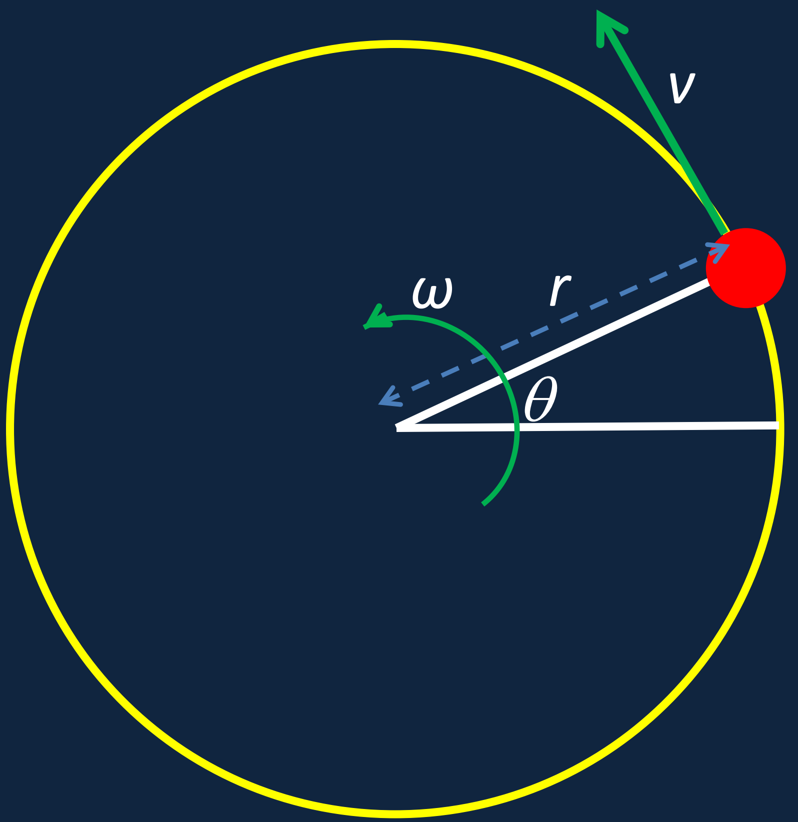
- Angular velocity: omega, $\omega = d\theta/dt$.

- Angular accleration: alpha, $\alpha = d\omega/dt = d^2\theta/dt^2$

Acceleration

- The tangential speed (along the rim) is $v = r\omega$, so the tangential acceleration is
- $a = dv/dt = r d\omega/dt = r\alpha$.
- The centripetal acceleration is

  $v^2/r = r\omega^2$.

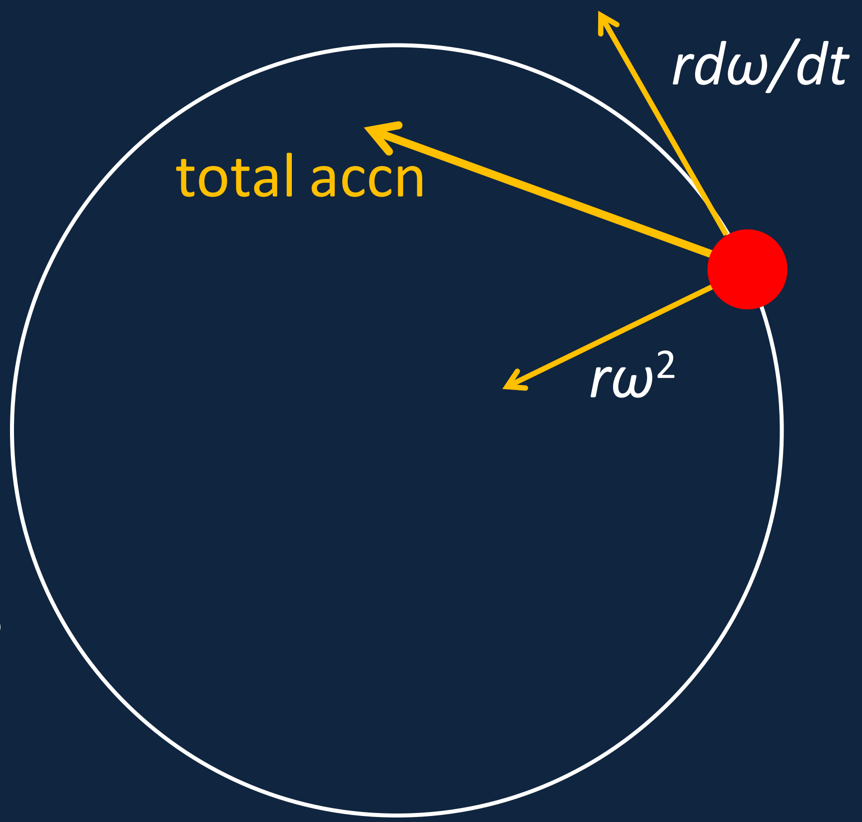


Components of Acceleration

- The tangential speed (along the rim) is $v = r\omega$, so the tangential acceleration (parallel to the rim) is

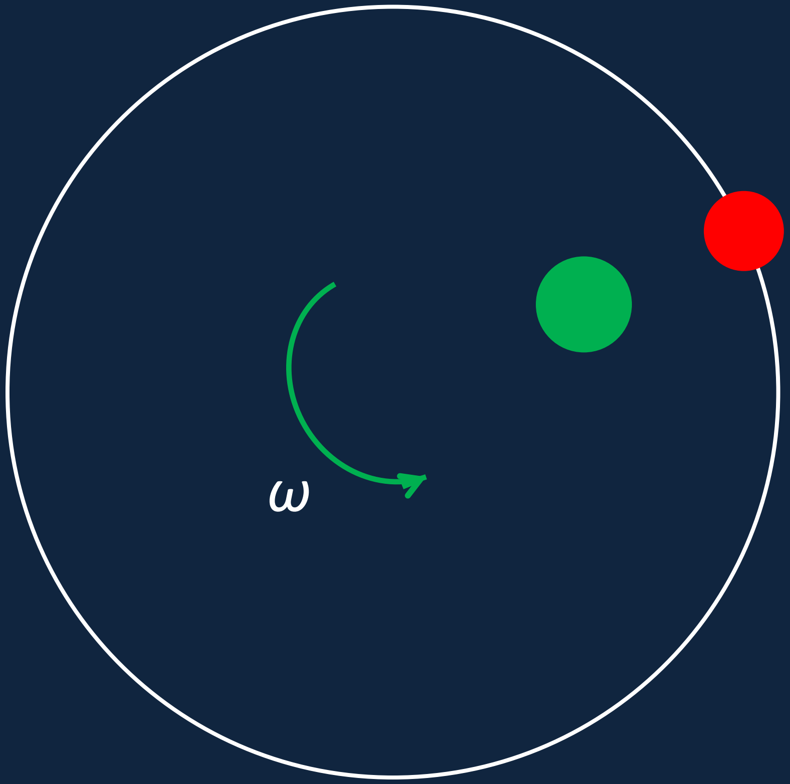
  $$dv/dt = r d\omega/dt = r\alpha.$$

- The centripetal acceleration is
- $$v^2/r = r\omega^2.$$
- Note: this formula is useful for comparing accelerations at different radii.



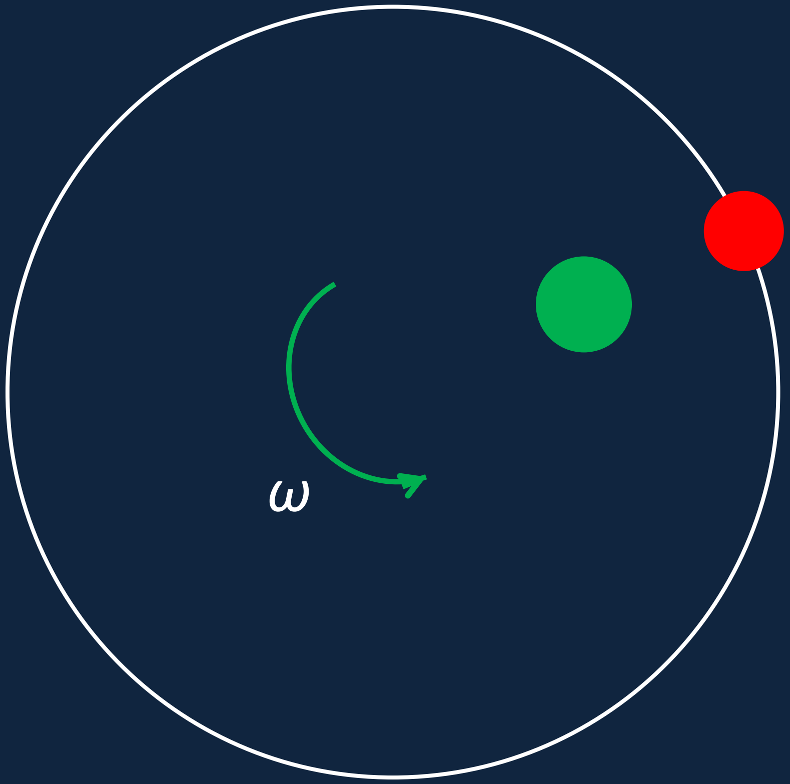
Clicker Question

- A red ball and a green ball are attached to a wheel as shown. The wheel is rotating at angular velocity ω, with <u>nonzero</u> angular acceleration α.
- Is the direction of total acceleration of the red ball parallel to that of the green ball?
- A Yes. B No.



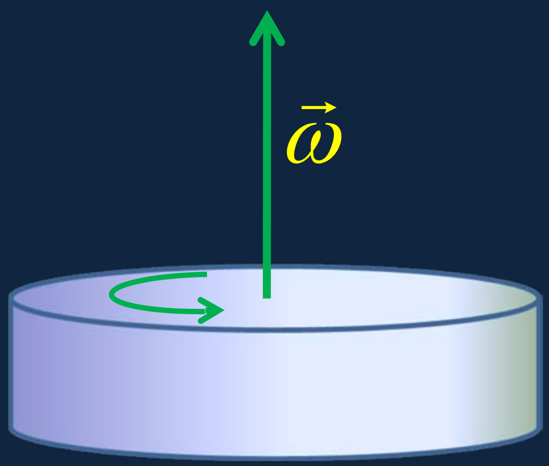
Clicker Answer

- A red ball and a green ball are attached to a wheel as shown. The wheel is rotating at angular velocity ω, with nonzero angular acceleration α.
- Is the direction of total acceleration of the red ball parallel to that of the green ball?
- A Yes. B No.
- The tangential acceleration of the red ball is $r\alpha$, its centripetal acceleration is $r\omega^2$.
- The green ball has the same values for the angular variables α and ω, so if it is at half the radius of the red ball, BOTH components of the acceleration are less by a factor of 2.



Angular Velocity as a Vector

- It will turn out to be essential later to represent angular velocity as a vector, with magnitude equal to the angular speed (radians per second) and direction along the axis of rotation.
- The convention, the "right hand rule" is given by curling up your right-hand fingers, your thumb pointing away from the palm, then if the fingers curl in the direction of rotation, the thumb is in the direction of $\vec{\omega}$.



Constant Angular Acceleration

- The formulas for angular velocity and position as functions of time for constant angular acceleration are precisely analogous to those for constant linear acceleration derived previously:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\left(\theta - \theta_0\right)$$

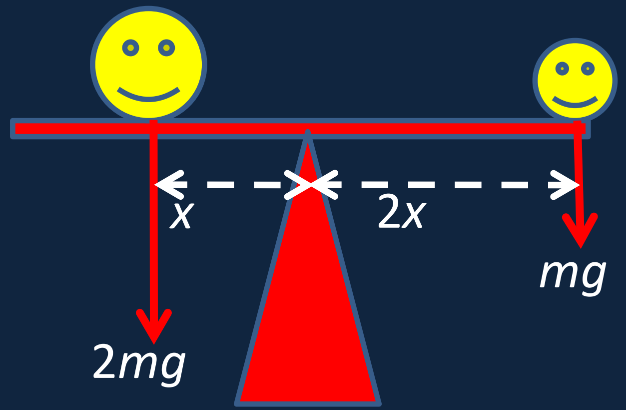
- Just be sure before you use these formulas that you really *do* have constant acceleration!

Torque

- The two kids shown have the same torque about the axle:
- **Torque = force x distance from the axle of the force's line of action.**
- Notation: torque

$$\tau = Fd = 2mgx$$

- Torque is also called "moment of a force" the distance *d* the "moment arm".

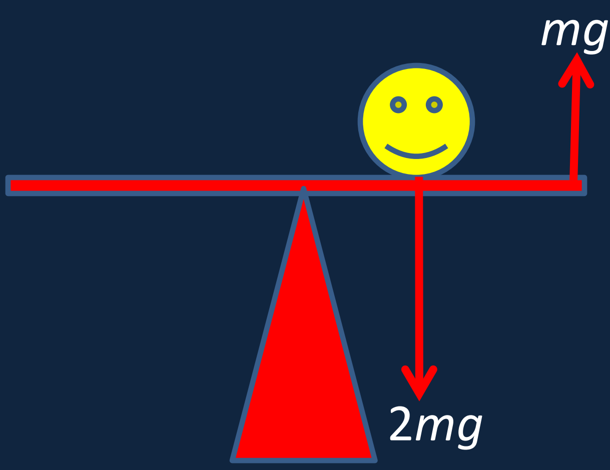


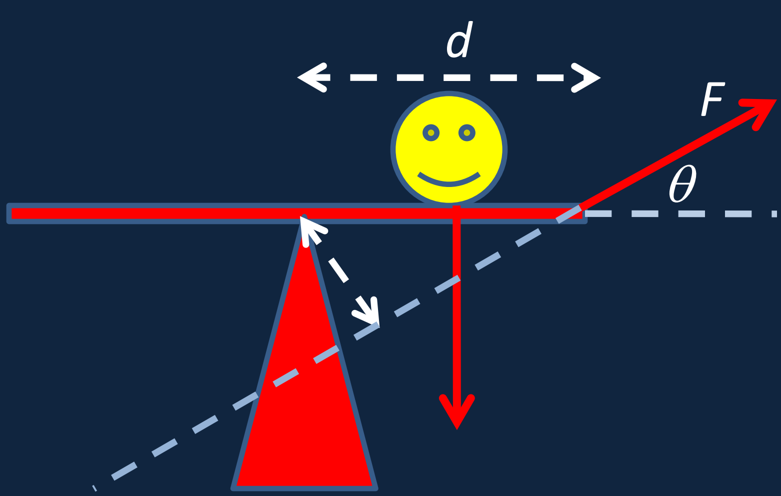
More Ways to Balance Torques…

- The two forces can act on the same side of the axle.
- The force does not need to be perpendicular to the lever arm: BUT only its component perpendicular to the arm exerts torque

$$\tau = Fd\sin\theta$$

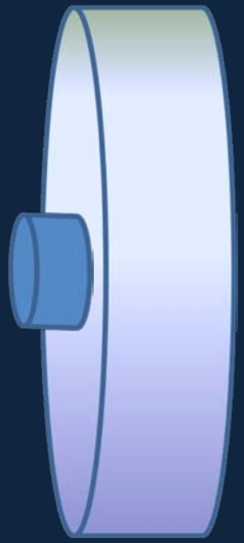
- Alternatively, one can draw the whole line of action of the force and find the perpendicular distance.





Rotational Dynamics

- Newton's First Law: a rotating body will continue to rotate at constant angular velocity as long as there is no torque acting on it.
- Picture a grindstone on a smooth axle.
- BUT the axle must be *exactly* at the center of gravity—otherwise gravity will provide a torque, and the rotation will not be at constant velocity!



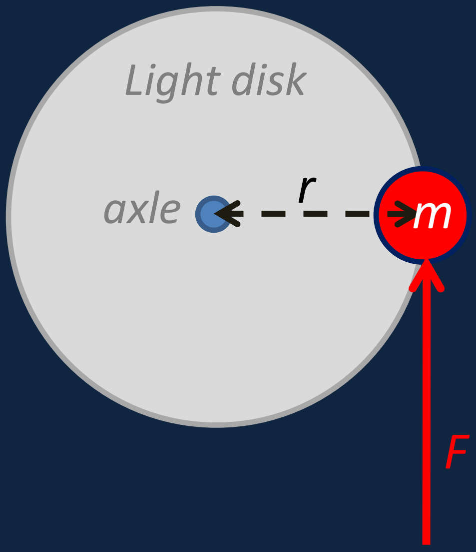
How is *Angular* Acceleration Related to Torque?

- Think about a tangential force F applied to a mass m attached to a light disk which can rotate about a fixed axis. (A *radially* directed force has zero torque, so does nothing.)
- The relevant equations are:

$$F = ma,\ a = r\alpha,\ \tau = rF.$$

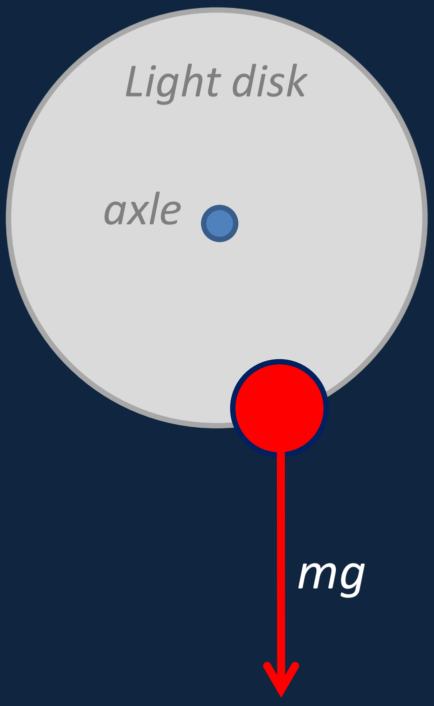
- Therefore $F = ma$ becomes

$$\tau = mr^2\alpha$$



Kinds of Equilibrium

- Suppose now the light disk is in a vertical plane, free to rotate about a horizontal axis.
- If the red mass is at rest at the lowest point, and is then displaced slightly, the torque from the gravitational force *mg* will pull it back towards the center. This is called stable equilibrium.
- The red mass can be at rest at the topmost point—but this is *unstable* equilibrium.
- If *g* = 0, we have neutral equilibrium.



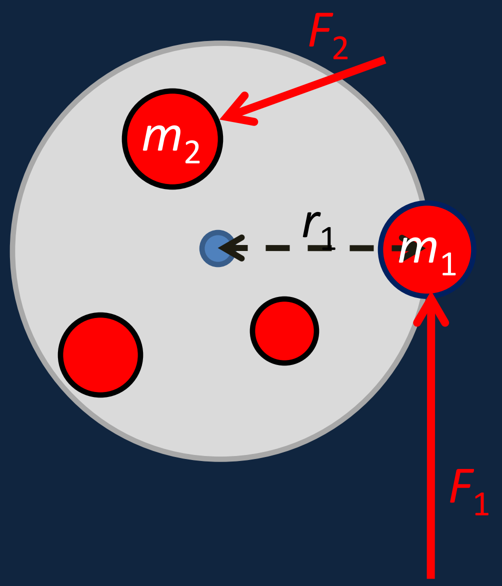
Newton's Second Law for Rotations

- For the special case of a mass m constrained by a light disk to circle around an axle, the angular acceleration α is proportional to the torque τ exactly as in the linear case the acceleration a is proportional to the force F.

- The angular equivalent of inertial mass m is the moment of inertia mr^2.

More Complicated Rotating Bodies

- Suppose now a light disk has several different masses attached at different places, and various forces act on them. As before, radial components cause no rotation, we have a sum of torques.
- BUT the rigid disk will cause a force on one mass to cause a torque on all the others! How do we handle *that*?



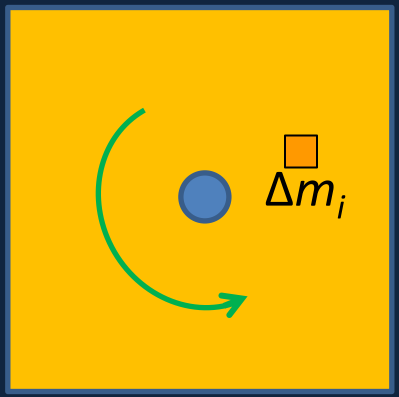
Newton's Third Law for a Rigid Rotating Body

- If a rigid body is made up of many masses m_i connected by rigid rods, the force exerted along the rod of m_i on m_j is equal in magnitude and opposite in direction to that of m_j on m_i, therefore the internal torques come in equal and opposite pairs, and therefore cannot contribute to the angular acceleration.
- It follows that the angular acceleration is generated by the sum of the external torques.

Moment of Inertia of a Solid Body

- Consider a flat square plate rotating about a perpendicular axis with angular acceleration α. One small part of it, Δm_i, distance r_i from the axle, has equation of motion

$$\tau_i = \tau_i^{\text{ext}} + \tau_i^{\text{int}} = \Delta m_i r_i^2 \alpha$$



- Adding contributions from all parts of the wheel

$$\tau = \sum_i \tau_i^{\text{ext}} = \left(\sum_i \Delta m_i r_i^2 \right) \alpha = I\alpha$$

- I is the Moment of Inertia.