Motion in Two and Three Dimensions: Vectors

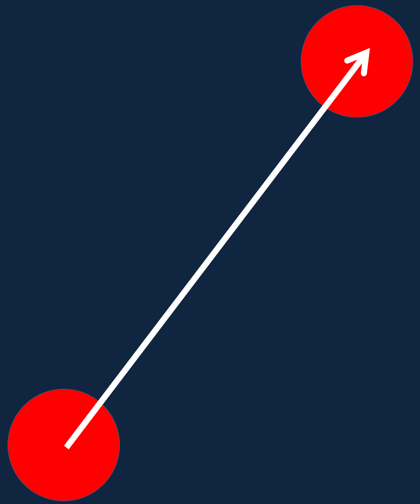
Physics 1425 Lecture 4

Today’s Topics

- In the previous lecture, we analyzed the motion of a particle moving vertically under gravity.
- In this lecture and the next, we’ll generalize to the case of a particle moving in two or three dimensions under gravity, like a projectile.
- First we must generalize displacement, velocity and acceleration to two and three dimensions: these generalizations are vectors.

Displacement

- We'll work usually in two dimensions—the three dimensional description is very similar.
- Suppose we move a ball from point A to point B on a tabletop. This displacement can be fully described by giving a distance and a direction.

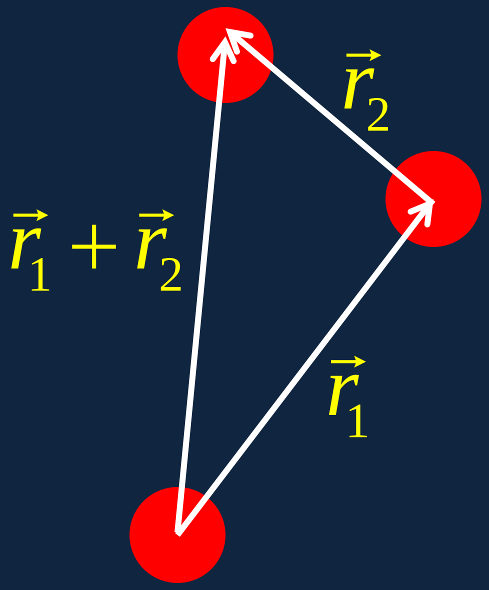


- Both can be represented by an arrow, the length some agreed scale: arrow length 10 cm representing 1 m displacement, say.
- This is a vector, written with an arrow $\vec{r}$: it has magnitude, meaning its length, written $|\vec{r}|$, and direction.

Displacement as a Vector

- Now move the ball a *second* time. It is evident that the total displacement , the sum of the two, called the resultant, is given by adding the two vectors tip to tail as shown:

- Adding displacement vectors (and notation!):



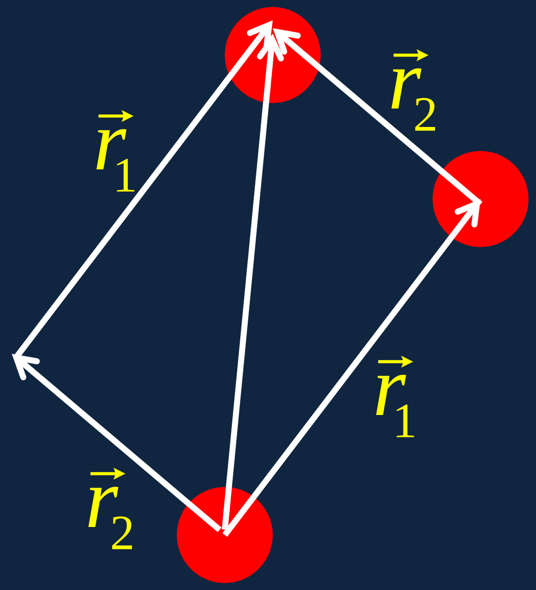
Adding Vectors

- You can see that

$$\vec{r}_1 + \vec{r}_2 = \vec{r}_2 + \vec{r}_1.$$

- The vector $\vec{r}_1$ represents a displacement, like saying walk 3 meters in a north-east direction: it works from any starting point.

- Adding vectors :



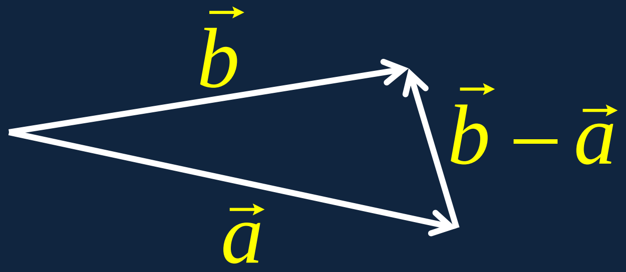
Subtracting Vectors

- It's pretty easy: just ask, what vector has to be added to $\vec{a}$ to get $\vec{b}$?
- The answer must be

$$\vec{b} - \vec{a}$$

- To construct it, put the ***tails*** of $\vec{a}$,$\vec{b}$ together, and draw the vector from the head of $\vec{a}$ to the head of $\vec{b}$.

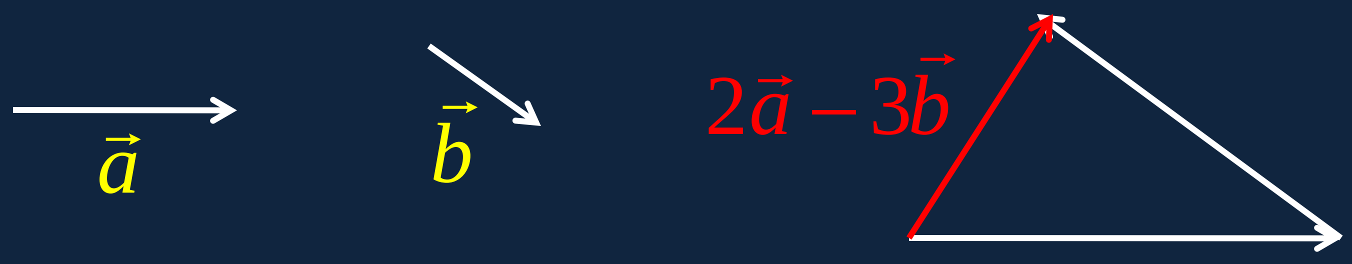
Finding the difference:



Multiplying Vectors by Numbers

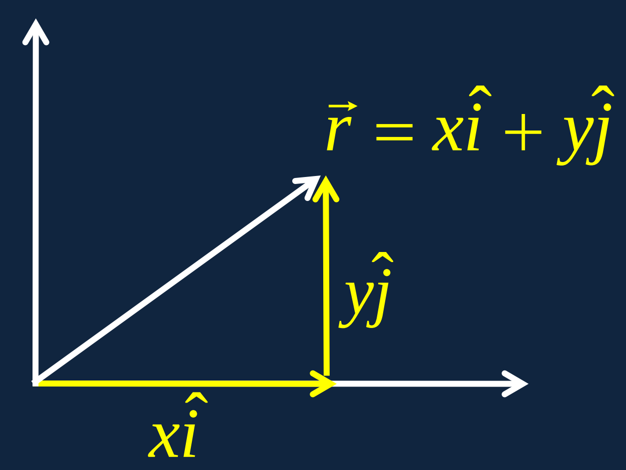
- Only the length changes: the direction stays the same.

- Multiplying and adding or subtracting:



Vector Components

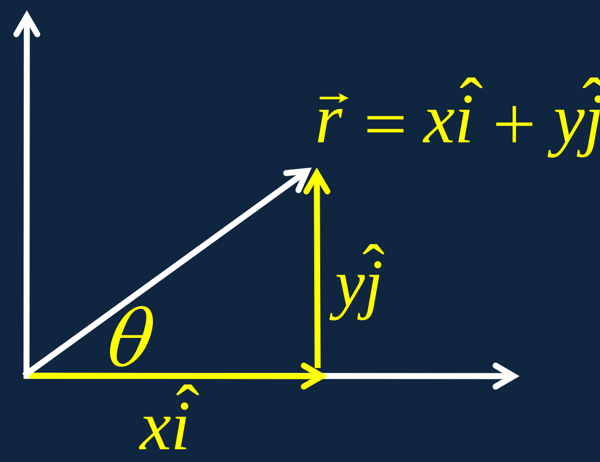
- Vectors can be related to the more familiar Cartesian coordinates (x, y) of a point P in a plane: suppose P is reached from the origin by a displacement $\vec{r}$.
- Then $\vec{r}$ can be written as the sum of successive displacements in the x- and y-directions:
- These are called the components of $\vec{r}$.
- Define $\hat{i}, \hat{j}$ to be vectors of unit length parallel to the x, y axes respectively. The components are $x\hat{i}, y\hat{j}$.



How $\vec{r}$ Relates to (x, y)

- The length (magnitude) of $\vec{r}$ is

$$|\vec{r}| = \sqrt{x^2 + y^2}$$



The angle between the vector and the x-axis is given by:

$$\tan\theta = \frac{y}{x}.$$

ConcepTest 3.1a Vectors I

If two vectors are given such that A + B = 0, what can you say about the magnitude and direction of vectors A and B?

1) same magnitude, but can be in any direction
2) same magnitude, but must be in the same direction
3) different magnitudes, but must be in the same direction
4) same magnitude, but must be in opposite directions
5) different magnitudes, but must be in opposite directions

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The magnitudes must be the same, but one vector must be pointing in the opposite direction of the other in order for the sum to come out to zero. You can prove this with the tip-to-tail method.

ConcepTest 3.1b Vectors II

Given that A + B = C, and that $|A|^2 + |B|^2 = |C|^2$, how are vectors A and B oriented with respect to each other?

1) they are perpendicular to each other
2) they are parallel and in the same direction
3) they are parallel but in the opposite direction
4) they are at 45° to each other
5) they can be at any angle to each other

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Note that the magnitudes of the vectors satisfy the Pythagorean Theorem. This suggests that they form a right triangle, with vector C as the hypotenuse. Thus, A and B are the legs of the right triangle and are therefore perpendicular.

ConcepTest 3.1c Vectors III

Given that A + B = C, and that |A| + |B| = |C| , how are vectors A and B oriented with respect to each other?

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2) **they are parallel and in the same direction**
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Given that A + B = C, and that |A| + |B| = |C|, how are vectors A and B oriented with respect to each other?

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The only time vector magnitudes will simply add together is when the direction does not have to be taken into account (i.e., the direction is the same for both vectors). In that case, there is no angle between them to worry about, so vectors A and B must be pointing in the same direction.

ConcepTest 3.3 Vector Addition

You are adding vectors of length 20 and 40 units. What is the only possible resultant magnitude that you can obtain out of the following choices?

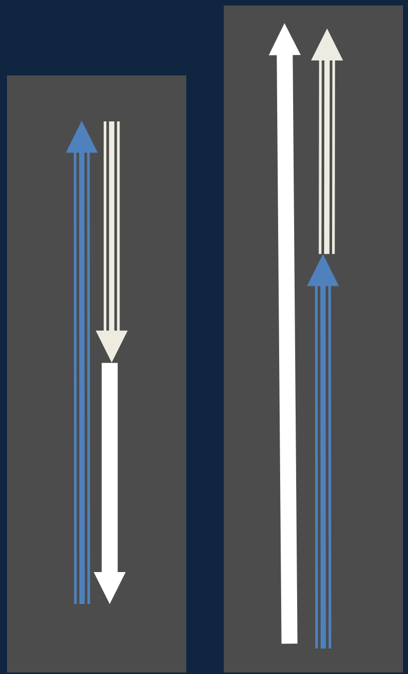
1) 0
2) 18
3) 37
4) 64
5) 100

ConcepTest 3.3 Vector Addition

You are adding vectors of length 20 and 40 units. What is the only possible resultant magnitude that you can obtain out of the following choices?

1) 0
2) 18
3) 37
4) 64
5) 100

The minimum resultant occurs when the vectors are opposite, giving 20 units. The maximum resultant occurs when the vectors are aligned, giving 60 units. Anything in between is also possible for angles between 0° and 180°.



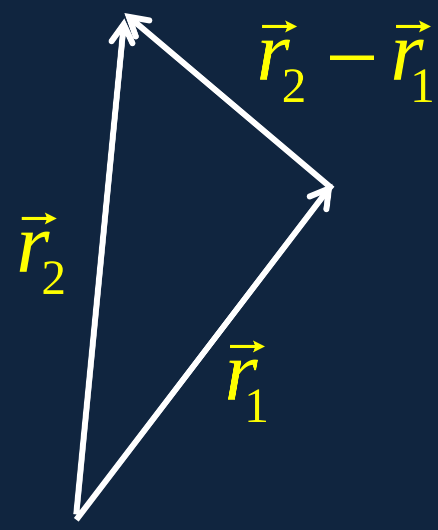
Average Velocity in Two Dimensions

average velocity = displacement/time

- A

In moving from point $\vec{r}_1$ to $\vec{r}_2$, the average velocity is in the direction $\vec{r}_2 - \vec{r}_1$:

$$\boxed{\bar{\vec{v}} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}}$$

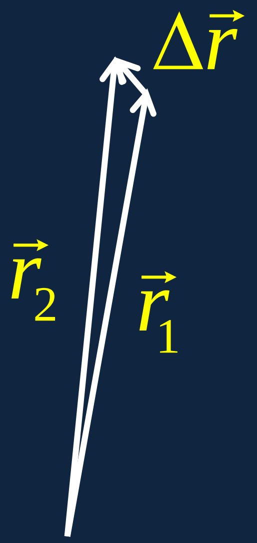


Instantaneous Velocity in Two Dimensions

$$\boxed{\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}}$$

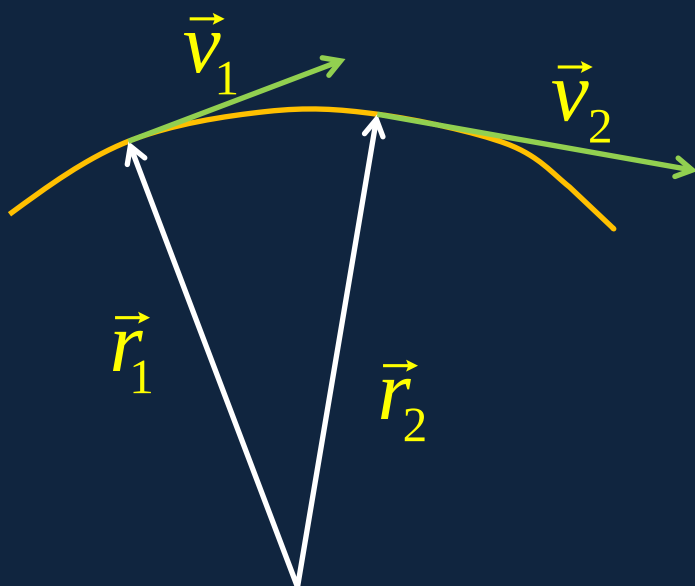
- Note: $\Delta \vec{r}$ is small, but that doesn't mean $\vec{v}$ has to be small— Δt is small too!

Defined as the average velocity over a vanishingly small time interval : points in direction of motion at that instant:

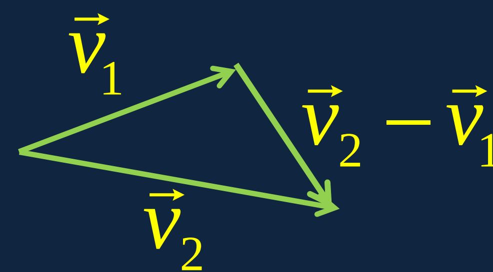


Average Acceleration in Two Dimensions

- Car moving along curving road:



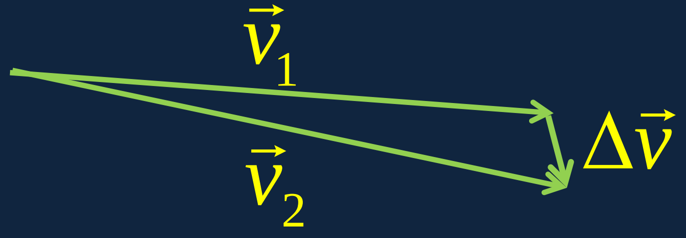
$$\overline{\vec{a}} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$



Note that the velocity vectors ***tails*** must be together to find the difference between them.

Instantaneous Acceleration in Two Dimensions

$$\boxed{\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}}$$



Acceleration in Vector Components

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}}$$

Writing $\vec{a} = \left(a_x, a_y\right),\ \vec{r} = \left(x, y\right)$ and matching:

$$a_x = \frac{d^2x}{dt^2}, \quad a_y = \frac{d^2y}{dt^2}$$

as you would expect from the one-dimensional case.

Clicker Question

A car is moving around a circular track at a constant speed. What can you say about its acceleration?

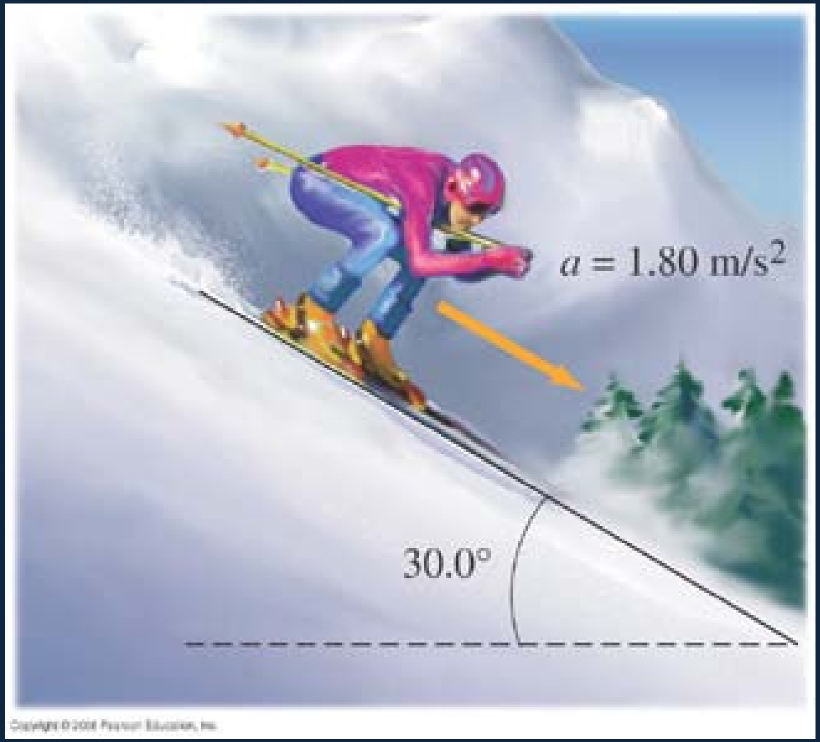
A. It’s along the track
B. It’s outwards, away from the center of the circle
C. It’s inwards
D. There is no acceleration

Sample Problem

- **15.** (II) The summit of a mountain, 2450 m above base camp, is measured on a map to be 4580 m horizontally from the camp in a direction 32.4° west of north. What are the components of the displacement vector from camp to summit? What is its magnitude? Choose the *x* axis east, *y* axis north, and *z* axis up.

Going down …

- **22.** (II) (*a*) A skier is accelerating down a 30.0° hill at 1.8 m/s (Fig. 3–39). What is the vertical component of her acceleration? (*b*) How long will it take her to reach the bottom of the hill, assuming she starts from rest and accelerates uniformly, if the elevation change is 325 m?

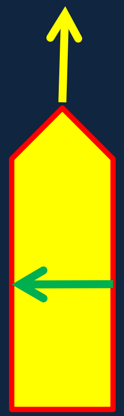


Relative Velocity

Running Across a Ship

- A cruise ship is going north at 4 m/s through still water.
- You jog at 3 m/s directly across

the ship from one side to the other.

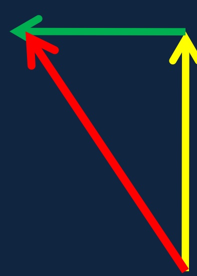


- What is your velocity *relative to the water*?

Relative Velocities Just Add…

- If the ship's velocity relative to the water is $\vec{v}_1$
- And your velocity relative to the water is $\vec{v}_2$
- Then your velocity relative to the water is

$$\boxed{\vec{v}_1 + \vec{v}_2}$$



- Hint: think how far you are *displaced* in one second!