More Angular Momentum

Physics 1425 Lecture 22

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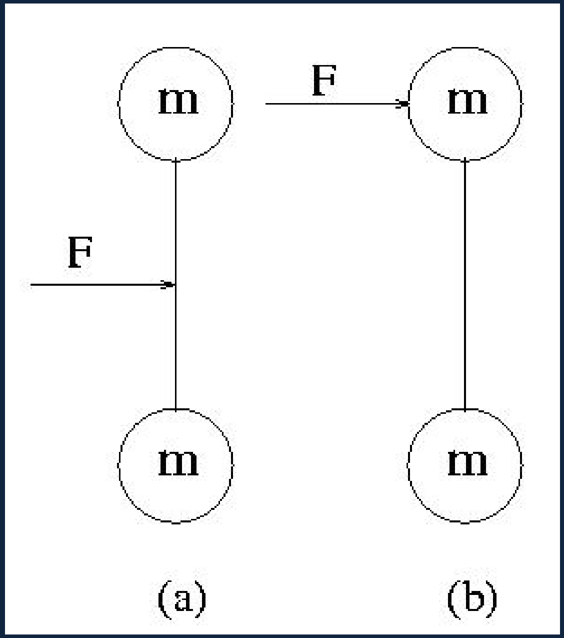
ConcepTest 10.9a Dumbbell I

A force is applied to a dumbbell for a certain period of time, first as in (a) and then as in (b). In which case does the dumbbell acquire the greater center-of-mass speed ?

1) case (a)
2) case (b)
3) no difference
4) it depends on the rotational inertia of the dumbbell



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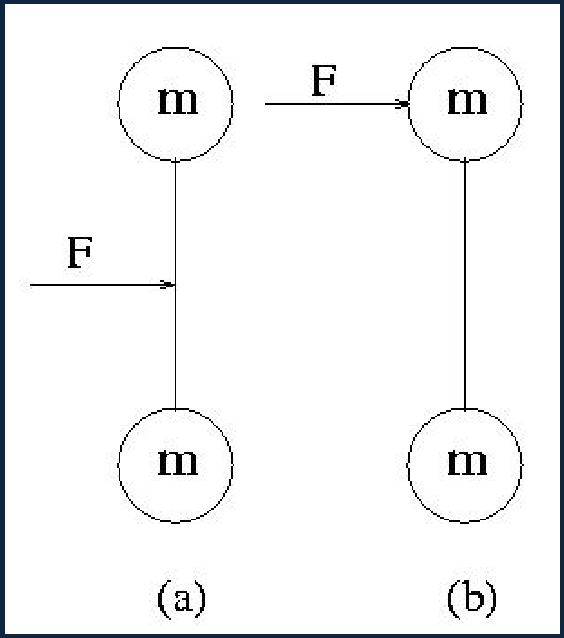

Because the same force acts for the same time interval in both cases, the change in momentum must be the same, thus the CM velocity must be the same.

ConcepTest 10.9b Dumbbell II

A force is applied to a dumbbell for a certain period of time, first as in (a) and then as in (b). In which case does the dumbbell acquire the greater energy ?

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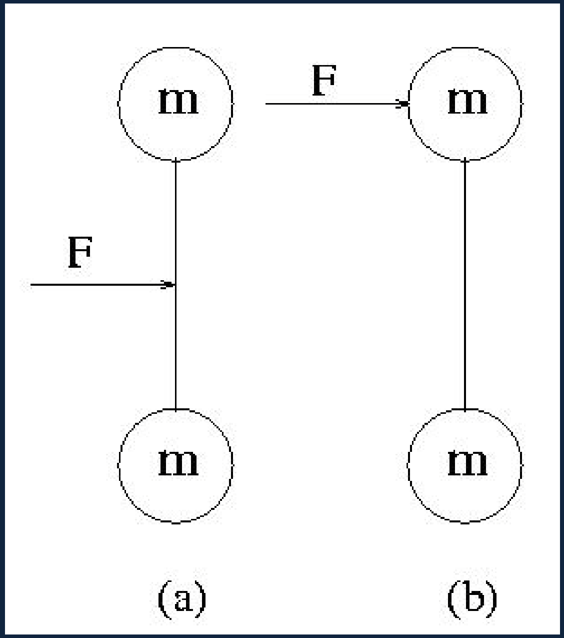

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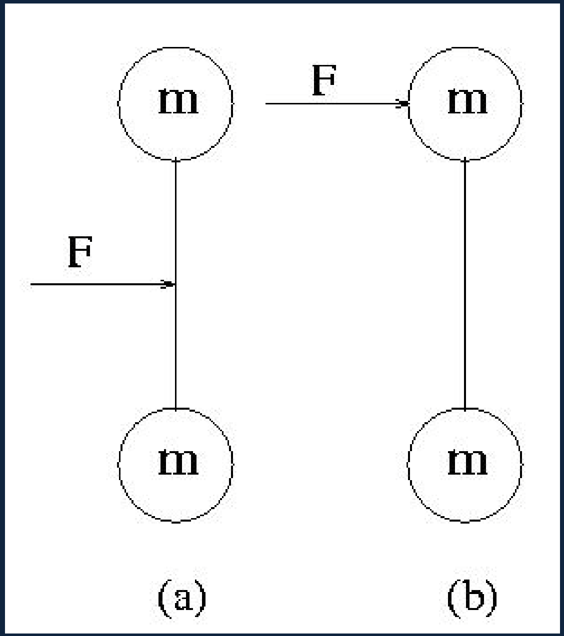
If the CM velocities are the same, the translational kinetic energies must be the same. Because dumbbell (b) is also rotating, it has rotational kinetic energy in addition.



Torque as a Vector

- Suppose we have a wheel spinning about a fixed axis: then $\vec{\omega}$ always points along the axis—so $d\vec{\omega}/dt$ points along the axis too.
- If we want to write a vector equation

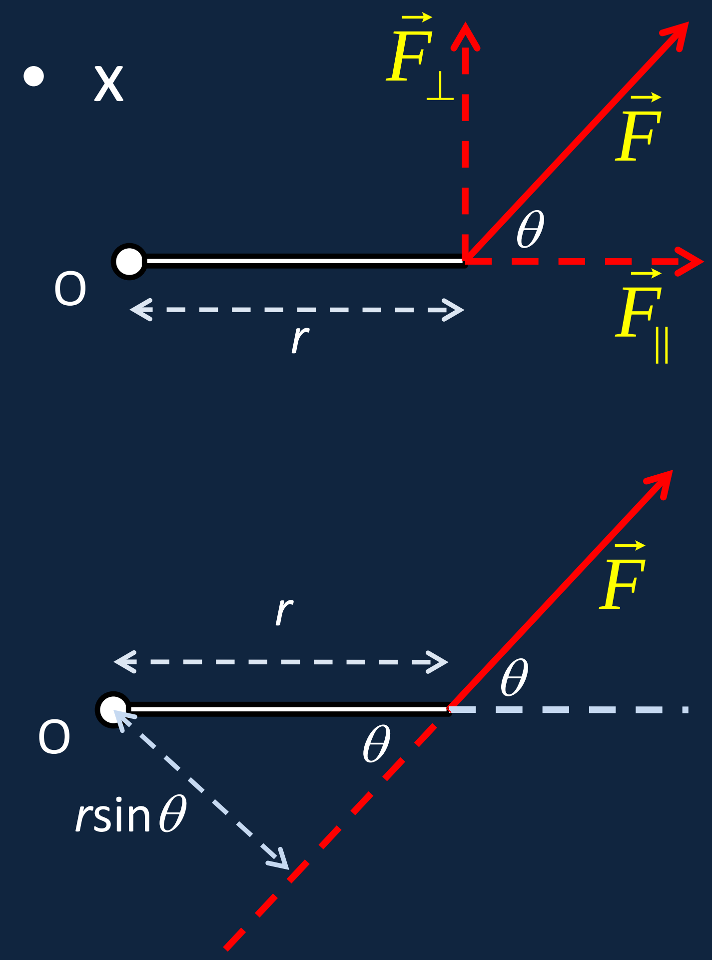
$$\vec{\tau} = I\vec{\alpha} = Id\vec{\omega}/dt$$

it's clear that the vector $\vec{\tau}$ is parallel to the vector $d\vec{\omega}/dt$: so $\vec{\tau}$ points along the axis too!

- BUT this vector $\vec{\tau}$, is, remember made of two other vectors: the force $\vec{F}$ and the place $\vec{r}$ where it acts!

More Torque…

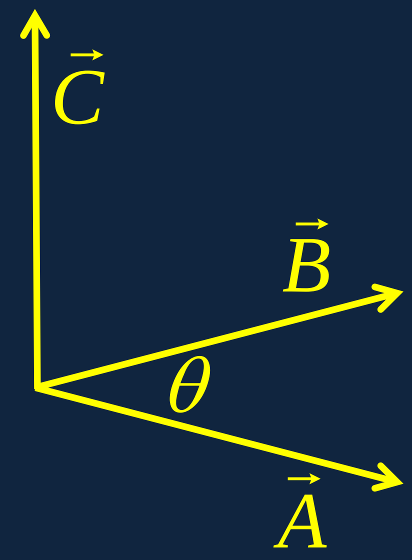
- Expressing the force vector $\vec{F}$ as a sum of components $\vec{F}_\perp$ ("fperp") perpendicular to the lever arm and $\vec{F}_\parallel$ parallel to the arm, it's clear that only $\vec{F}_\perp$ has leverage, that is, torque, about O.
  $\vec{F}_\perp$ has magnitude $F\sin\theta$, so $\tau = rF\sin\theta$.
- Alternatively, keep $\vec{F}$ and measure *its* lever arm about O: that's $r\sin\theta$.



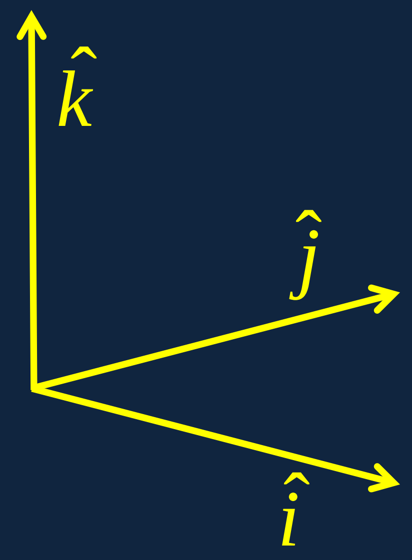
Definition: The Vector Cross Product

$$\vec{C} = \vec{A} \times \vec{B}$$

- The magnitude C is $AB\sin\theta$, where θ is the angle between the vectors $\vec{A}, \vec{B}$.
- The direction of $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$, and is your right thumb direction if your curling fingers go from $\vec{A}$ to $\vec{B}$.



The Vector Cross Product in Components



- Recall we defined the unit vectors $\hat{i}, \hat{j}, \hat{k}$ pointing along the *x*, *y*, *z* axes respectively, and a vector can be expressed as $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$
- Now $\hat{i}\times\hat{i} = 0,\ \hat{i}\times\hat{j} = \hat{k},\ \hat{i}\times\hat{k} = -\hat{j},\ldots$
- So

$$\vec{A}\times\vec{B} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\times\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

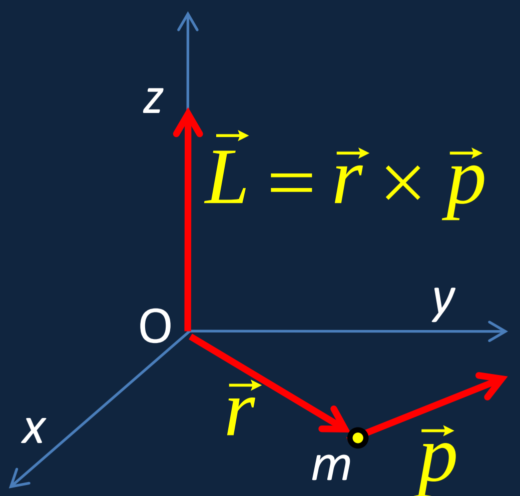
$$= \hat{i}\left(A_yB_z - A_zB_y\right) + \ldots$$

Vector Angular Momentum of a Particle

- A particle with momentum $\vec{p}$ is at position $\vec{r}$ from the origin O.
- Its angular momentum about the origin is

$$\vec{L} = \vec{r} \times \vec{p}$$

- This is in line with our definition for part of a rigid body rotating about an axis: *but also works for a particle flying through space*.



Viewing the *x*-axis as coming out of the slide, this is a “right-handed” set of axes: $\hat{i} \times \hat{j} = +\hat{k}$

Angular Momentum and Torque for a Particle

- Angular momentum about the origin of particle mass *m*, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

Torque about the origin

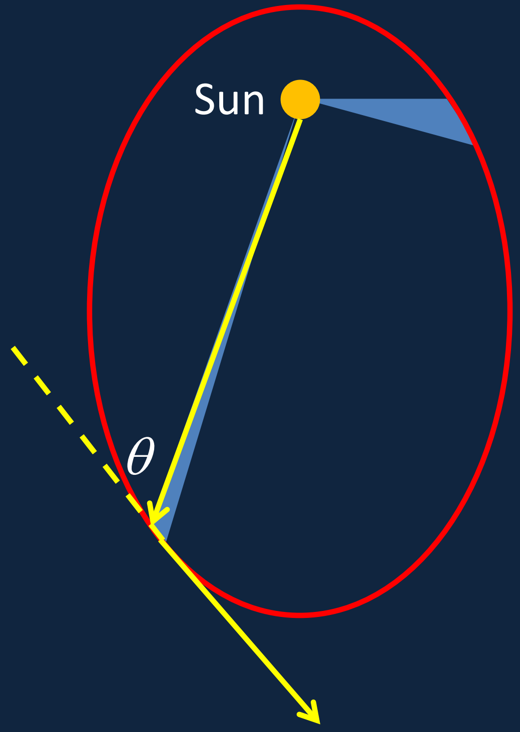
- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

Kepler's Second Law

As the planet moves, a line from the planet to the center of the Sun sweeps out equal areas in equal times.

- In unit time, it moves through a distance $\vec{v}$.
- The area of the triangle swept out is $\frac{1}{2} r v \sin\theta$ (from ½ base x height)
- This is $\frac{1}{2} L/m$, $\vec{L} = \vec{r} \times \vec{p}$.
- Kepler's Law is telling us the angular momentum about the Sun is constant: this is because the Sun's pull has *zero torque* about the Sun itself.



The base of the thin blue triangle is a distance *v* along the tangent. The height is the perp distance of this tangent from the Sun.

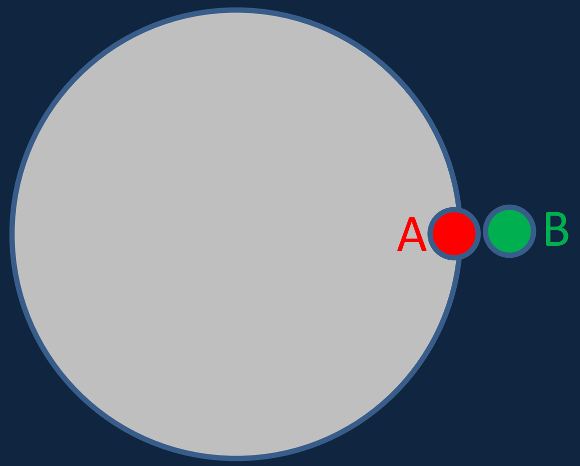
Guy on Turntable

- A, of mass m, is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R, next to B, who's on the ground.
- A now walks around the edge until he's back with B.
- How far does he walk?

A. $2\pi R$

B. $2.5\pi R$

C. $3\pi R$



Guy on Turntable: Answer

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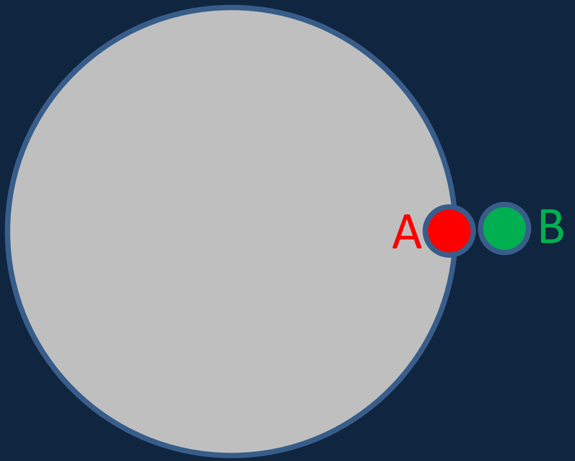
  $3\pi R$

His moment of inertia is mR^2, the turntable's is $2mR^2$. There is zero total angular momentum, so if he walks around with angular velocity ω relative to the ground, the turntable has angular velocity $-\omega/2$. If he marked the turntable at the point he began, he'd reach that mark again after walking 2/3rds of the way round, as the turntable turned the other way to meet him. When he gets back to B, the turntable has done half a complete turn.



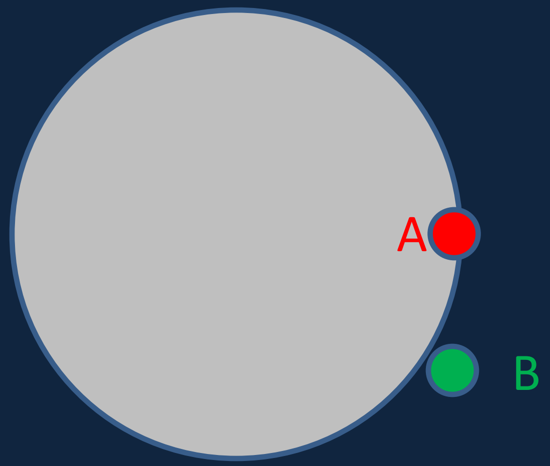
Guy on Turntable Catches a Ball

- A, of mass *m*, is standing on the edge of a frictionless turntable, a disk of mass 4*m*, radius *R*, at rest.
- B, who's on the ground, throws a ball weighing 0.1*m* at speed *v* to A, who catches it without slipping.
- What is the angular momentum of turntable + man + ball now?

A. 0.1mvR

B. (0.1/3.1)mvR

C. (0.1/5.1)mvR



On the Ball? Answer

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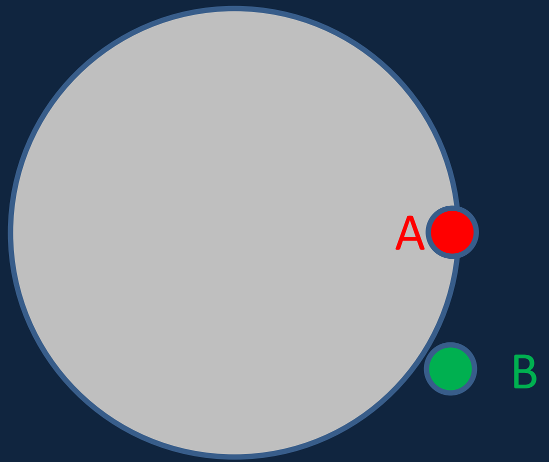

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The ball thrown from B to A is moving in the direction of the tangent at A, the angular momentum about a point of a particle flying through the air equals $\vec{r} \times m\vec{v}$ and the line of the velocity is perp to the radius ending at A, so the angular momentum of the ball about the disk center is 0.1*mvR*. There is no other angular momentum, so this is shared with the man and the turntable.

Guy on Turntable Walks In

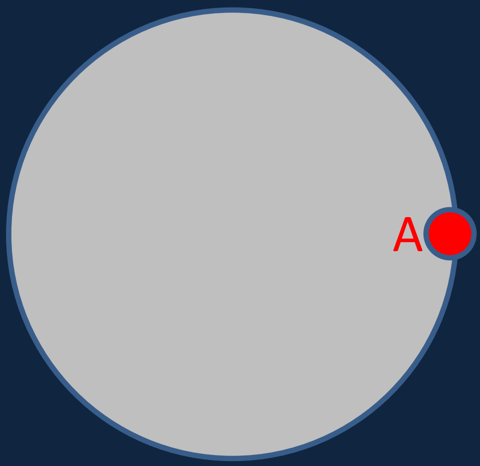
- A, of mass *m*, is standing on the edge of a frictionless turntable, a disk of mass 4*m*, radius *R*, which is rotating at 6 rpm.
- A walks to the exact center of the turntable.
- How fast (approximately) is the turntable now rotating?

A. 12 rpm

B. 9 rpm

C. 6 rpm

D. 4 rpm



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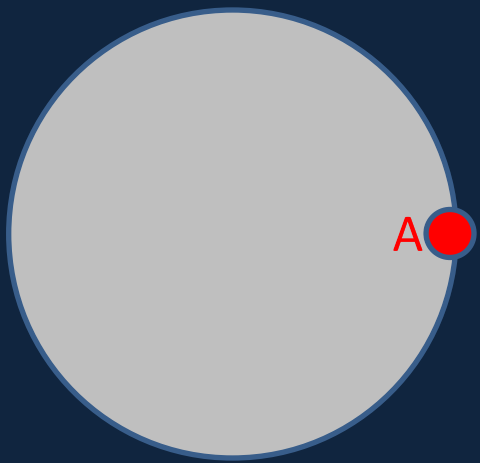


Initially, the man has moment of inertia mR^2, the turntable $2mR^2$. Finally, the man has negligible moment of inertia, so the total I decreases by a factor of 2/3, to conserve angular momentum (ther are no external torques) ω increases by 3/2.

Reminder: Angular Momentum and Torque for a Particle…

- Angular momentum about the origin of particle mass *m*, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

Lots of Particles

- Suppose we have particles acted on by external forces, and also acting on each other.
- The rate of change of angular momentum of one of the particles about a fixed origin O is:

$$d\vec{L}_i / dt = \vec{\tau}_{i\ \mathrm{int}} + \vec{\tau}_{i\ \mathrm{ext}}$$

- The internal torques come in equal and opposite pairs, so

$$d\vec{L} / dt = \sum_i d\vec{L}_i / dt = \sum_i \vec{\tau}_{i\ \mathrm{ext}}$$

Rotational Motion of a Rigid Body

- For a collection of interacting particles, we've seen that

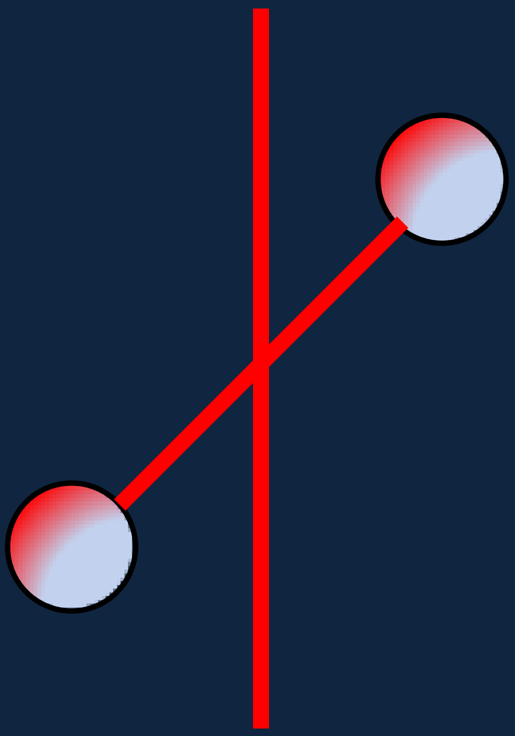
$$d\vec{L} / dt = \sum_i \vec{\tau}_i$$

  the vector sum of the applied torques, $\vec{L}$ and the $\vec{\tau}_i$ being measured about a fixed origin O.
- A rigid body is equivalent to a set of connected particles, so the same equation holds.
- It is also true (proof in book) that even if the CM is accelerating,

$$d\vec{L}_{\mathrm{CM}} / dt = \sum \vec{\tau}_{\mathrm{CM}}$$

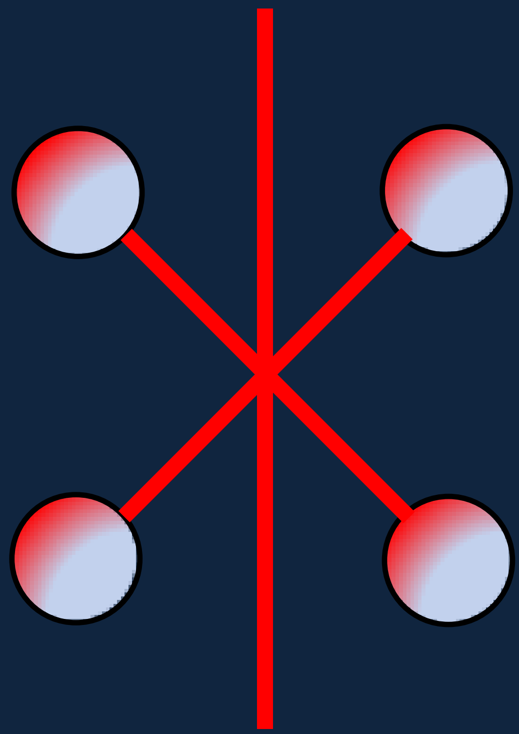
Angular Velocity and Angular Momentum Need not be Parallel

- Imagine a dumbbell attached at its center of mass to a light vertical rod as shown, then the system rotates about the vertical line.
- The angular velocity vector $\vec{\omega}$ is vertical.
- The total angular momentum $\vec{L}$ about the CM is $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$.
- Think about this at the instant the balls are in the plane of the slide—so is $\vec{L}$, but it's not vertical!



When *are* Angular Velocity and Angular Momentum Parallel?

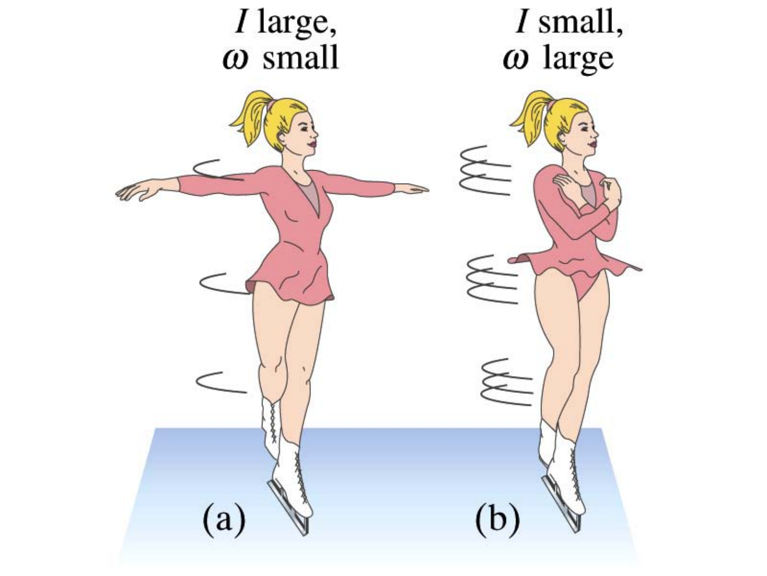
- When the rotating object is symmetric about the axis of rotation: if for each mass on one side of the axis, there's an equal mass at the corresponding point on the other side.
- For this pair of masses, $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$ *is* along the axis.
- (Check it out!)



ConcepTest 11.1 Figure Skater

A figure skater spins with her arms extended. When she pulls in her arms, she reduces her rotational inertia and spins faster so that her angular momentum is conserved. Compared to her initial rotational kinetic energy, her rotational kinetic energy after she pulls in her arms must be

1) the same
2) larger because she's rotating faster
3) smaller because her rotational inertia is smaller



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$KE_{rot} = \frac{1}{2} I \omega^2 = \frac{1}{2} L \omega$ (used $L = I\omega$). Because L is conserved, larger ω means larger KE_{rot}. The "extra" energy comes from the work she does on her arms.