More Rotational Dynamics

Physics 1425 Lecture 20

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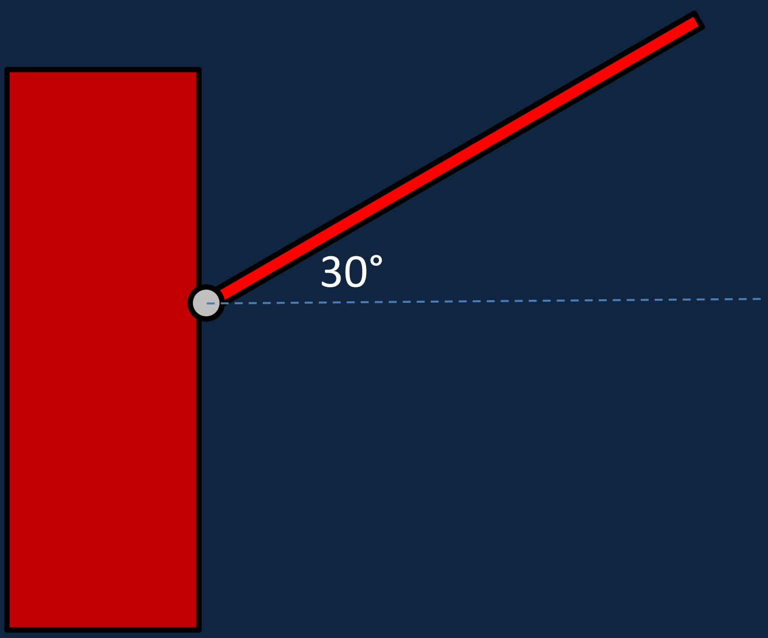
Clicker Question

A uniform rod is free to rotate in a vertical plane about a frictionless hinge at one end. It is released from rest at an angle of 30°. ($I = (1/3)ML^2$, $\tau = Mg(L/2)\cos 30°$) The initial downward acceleration of the free end of the rod is:



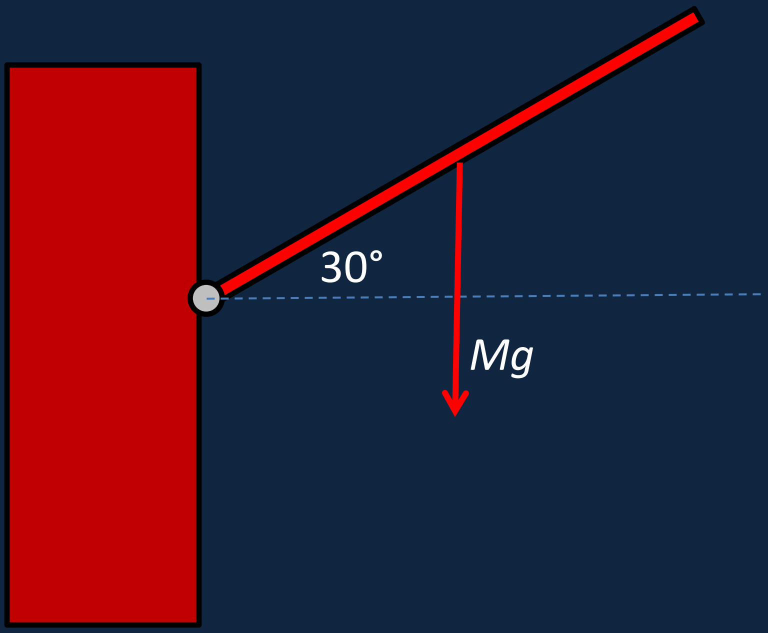
A. equal to g

B. greater than g

C. less than g

Clicker Answer

It's *greater* than g! The moment of inertia about the hinge is $(1/3)ML^2$, the torque is $(MgL/2)\cos 30°$, so the acceleration is given by $\tau = I\alpha$, $\alpha = (3g/2L)\cos 30°$, the far end accelerates at $L\alpha = (3g/2)\cos 30° > g$.



Ball in cup video

Falling coins

Rotational Kinetic Energy

- Imagine a rotating body as composed of many small masses m_i at distances r_i from the axis of rotation.
- The mass m_i has speed $v = \omega r_i$, so $KE = ½m_i r_i^2 \omega^2$.
- The total *KE* of the rotating body (assuming the axis is at rest) is

$$\boxed{K = \sum_i \left(\tfrac{1}{2} m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I \omega^2}$$

Problem from Book

- **69.** A 2.30-m-long pole is balanced vertically on its tip. It starts to fall and its lower end does not slip. What will be the speed of the upper end of the pole just before it hits the ground? [*Hint*: Use conservation of energy.]

Torque Power

- If a net torque τ is acting on a rotating body, the net power is the rate of change of rotational energy

$$\frac{d}{dt}\left(\tfrac{1}{2}I\omega^2\right) = I\omega\frac{d\omega}{dt} = I\omega\alpha = \omega\tau \;\; (\text{recall } \; \tau = I\alpha)$$

- So the rate of working of the torque,

  power = $\tau\omega$, its value x the angular velocity.
- Total work done over some time period is

$$\int \tau\omega dt = \int \tau\frac{d\theta}{dt}dt = \int \tau d\theta$$

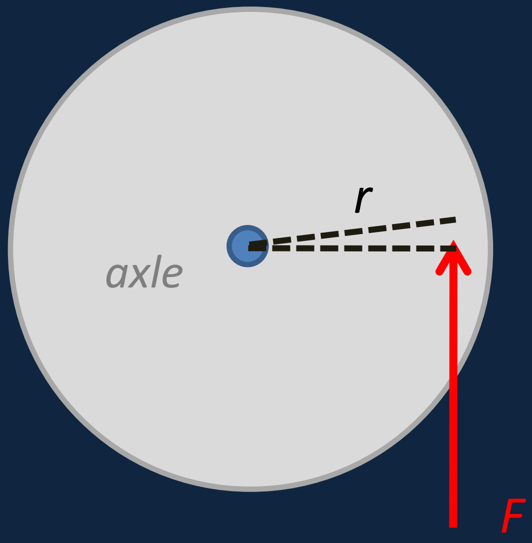
- This is just like $\int F dx$ in linear motion.

Work Done by a Torque

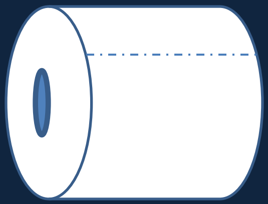
- Suppose the torque is a force F acting at a distance r from the center as shown. If the disk turns through an angle $d\theta$, the force acts through a distance $ds = rd\theta$ so does work $Fds = Frd\theta$.
- But $\tau = rF$, so the work

$$Fds = Frd\theta = \tau d\theta$$

Force x distance = torque x angle



A Familiar Item…



- A roll of toilet paper has diameter 0.1m, which happens also to be the length of one sheet.
- What is the angle *in radians* subtended at the central line of the roll by one sheet in the outside layer?
- A. 1

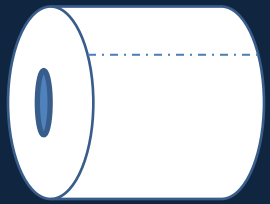
  B. 2

  C. 0.5

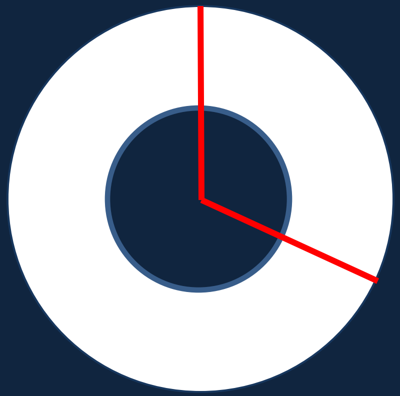
  D. π

  E. 1/π

A Familiar Item…



- A roll of toilet paper has diameter 0.1m, which happens also to be the length of one sheet.
- What is the angle *in radians* subtended at the central line of the roll by one sheet in the outside layer?
- It's about 2 radians:



On a Roll…

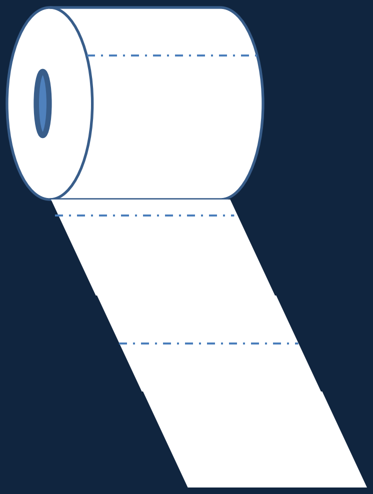
- This roll (0.1 m diameter, 0.1 m sheets) rolls across the table, unwinding three sheets per second.
- Give its CM velocity, *and* the angular velocity about the CM in radians/sec.

A. 0.3, 6

B. 0.3, 3

C. 0.6, 6

D. 0.3, 3π



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A. 0.3, 6

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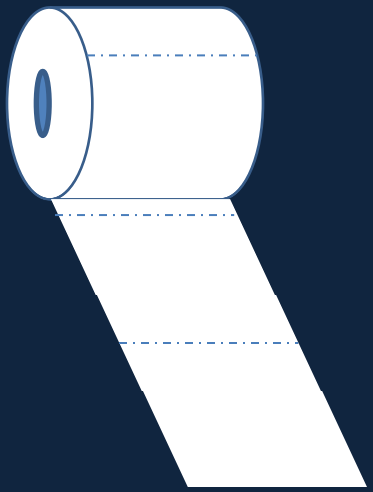
C. 0.6, 6

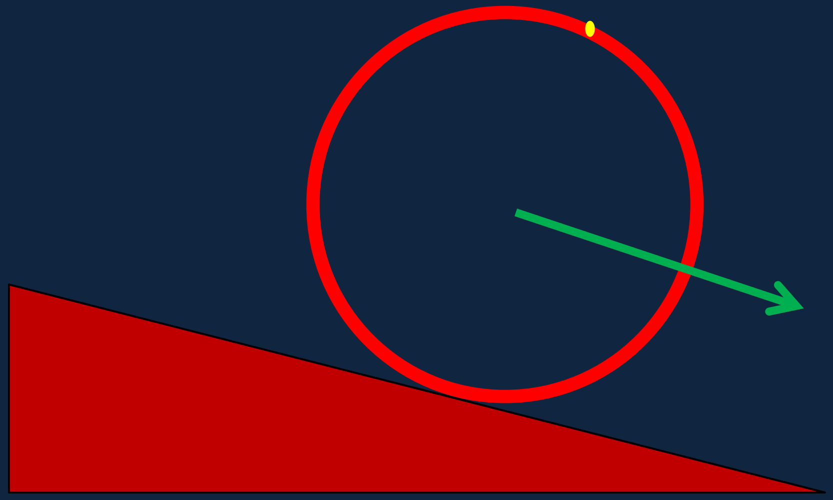
D. 0.3, 3π

Remember $\omega = vr$, and three sheets in one second is 6 radians—almost a complete revolution.



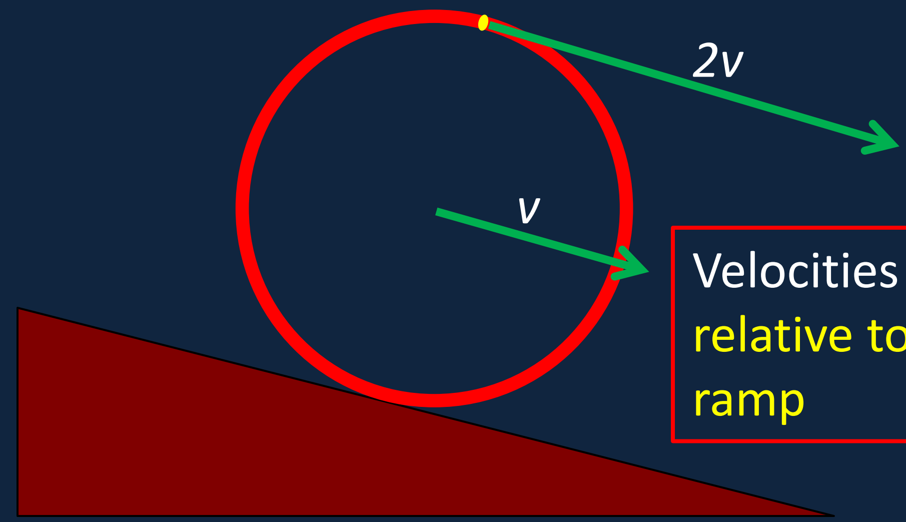
Clicker Question

- A hoop is rolling down a ramp (without slipping) at v m/sec.
- How fast is the point on the hoop furthest from the ramp moving?
- A. v m/sec
- B. $2v$ m/sec
- C. $4v$ m/sec

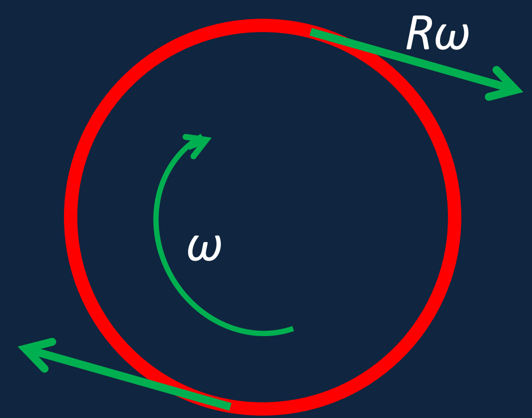


Hoop Rolling Down Ramp

- If there's no slipping, the point on the hoop in contact with the ramp is at *rest*—*the hoop is at that instant rotating about that point.*
- So if the center is moving at v, the "top" point is moving at $2v$.
- Relative to the center, all points are moving at speed $R\omega$ tangentially.
- Hence, since the bottom's at rest: $\boxed{v = R\omega}$
- The "<u>no slip</u>" condition.



Velocities relative to ramp



Velocities relative to center of hoop

Total Kinetic Energy of Rolling Hoop

- Suppose as usual the hoop is made of many small masses m_i and the mass m_i is moving at $\vec{v}_i$. Then the total KE is $\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}$.
- This total kinetic energy depends on both the translational motion (the center of the hoop is moving) and the hoop's rotation about the center.
- How do we sort this out?

Separating Translational and Rotational Kinetic Energies: Details

- Suppose we have rigid body we represent as a collection of masses m_i, with individual velocities $\vec{v}_i$.
- Let's suppose the CM is moving at $\vec{v}_{\mathrm{CM}}$, so the total linear momentum is $M\,\vec{v}_{\mathrm{CM}}$, M being the total mass.
- To separate out the rotational motion, we'll write the individual velocities $\vec{v}_i = \vec{v}_{\mathrm{CM}} + \vec{u}_i$: so $\vec{u}_i$ is velocity of m_i relative to the CM.
- Then the total kinetic energy is

$$\sum_i \tfrac{1}{2} m_i \vec{v}_i^{\,2} = \sum_i \tfrac{1}{2} m_i \left(\vec{v}_{\mathrm{CM}} + \vec{u}_i\right)^2 = \tfrac{1}{2} M \vec{v}_{\mathrm{CM}}^{\,2} + \vec{v}_{\mathrm{CM}} \cdot \sum_i m_i \vec{u}_i + \sum_i \tfrac{1}{2} m_i \vec{u}_i^{\,2}$$

$$\boxed{KE = \tfrac{1}{2} M \vec{v}_{\mathrm{CM}}^{\,2} + \tfrac{1}{2} I_{\mathrm{CM}} \omega^2}$$

- Because relative to the CM $\sum_i m_i \vec{u}_i = \dfrac{d}{dt} \sum_i m_i \vec{r}_i = 0, \quad \vec{u}_i^{\,2} = r_i^2 \omega^2.$

Total Energy: the Bottom Line

- In case the last slide was too much, what you *really* need is that the total kinetic energy of a moving, rotating object is a sum of two terms:
- Translational KE, the same as if all the mass is moving with the velocity of the center of mass, and
- Rotational KE, about the center of mass:

$$\boxed{KE = \tfrac{1}{2} M \vec{v}_{\mathrm{CM}}^{2} + \tfrac{1}{2} I_{\mathrm{CM}} \omega^{2}}$$

How Fast Does a Hoop Roll Down a Ramp?

- Assuming no slipping, so

  $$v = R\omega$$

- The total kinetic energy at an instant:

  $KE = ½mv^2 + ½I\omega^2$

  $= ½mv^2 + ½(mR^2)\omega^2$

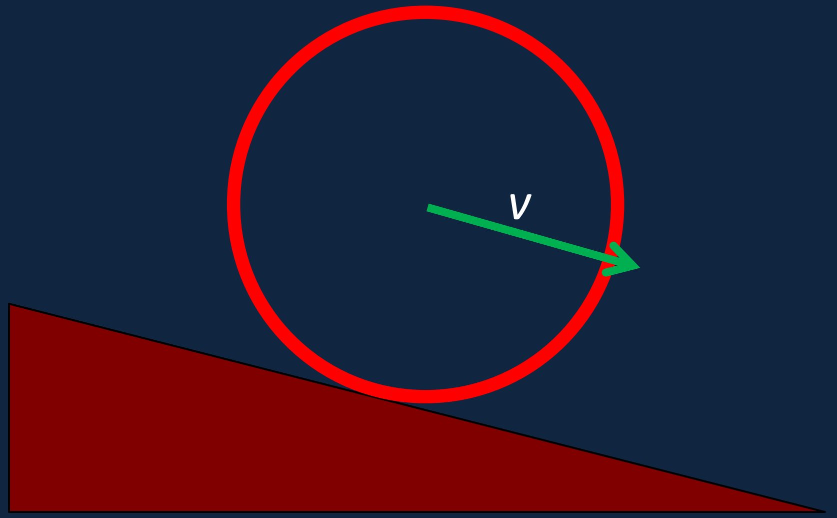
  $= mv^2$.

- If it's rolled down through height h from a standing start,

  $mv^2 = mgh$, so $v = \sqrt{(gh)}$

- For a frictionless sliding mass,

  $½mv^2 = mgh$, so $v = \sqrt{(2gh)}$: faster!



The hoop takes longer to get down than a low-friction sliding block, because the same loss in potential energy has to supply BOTH translational *KE* and rotational *KE* for the hoop.

Ramp Race

A hoop, a solid cylinder and a solid sphere roll down the same ramp from a standing start.   Who clocks the fastest time?

A. The hoop
B. The solid cylinder
C. The solid sphere
D. It depends on the sizes and/or masses.

Ramp Race

A hoop, a solid cylinder and a solid sphere roll down the same ramp from a standing start. Who clocks the fastest time?

The sphere wins: its mass is on average closer to the axis of rotation, so it has less rotational *KE* compared with translational *KE*.

A. The hoop
B. The solid cylinder
C. The solid sphere
D. It depends on the sizes and/or masses.

Note: for the sphere $I = (2/5)mR^2$ solid cylinder $\frac{1}{2}mR^2$, hoop mR^2.

A New Look for $\tau = I\alpha$

- We've seen how $\tau = I\alpha$ works for a body rotating about a fixed axis.
- <u>$\tau = I\alpha$ *i*s not true in general</u> if the axis of rotation is *itself* accelerating
- BUT it IS true if the axis is through the CM, and isn't changing direction!
- This is quite tricky to prove—it's in the book
- And $\boxed{\tau_{CM} = I_{CM}\alpha_{CM}}$ is often useful, as we'll see.