Maxwell's Equations

Physics 2415 Lecture 29

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Today's Topics

- Maxwell's equations
- The speed of light

Equations for Electricity and Magnetism

• <u>Gauss' law</u> for <u>electric</u> fields

$$\int \vec{E} \cdot \vec{dA} = q / \varepsilon_0$$

the electric flux out of a volume = (charge inside)/ ε_0 .

• Gauss' law for magnetic fields

$$\int \vec{B} \cdot \vec{dA} = 0$$

• There is no such thing as magnetic charge: magnetic field lines just *circulate*, so for any volume they flow out of, they flow back into it somewhere else.

Equations for Electricity and Magnetism

• Electrostatics: (no changing fields)

$$\oint \vec{E} \cdot \vec{d\ell} = 0$$

around any closed curve: this means the work done against the electric field from A to B is independent of path, the field is conservative: a potential energy can be defined.

• <u>Faraday's law of induction</u>: in the presence of a changing magnetic field, the above equation becomes:

$$\oint \vec{E} \cdot \vec{d\ell} = -d / dt \left(\int \vec{B} \cdot \vec{dA} \right) = -d\Phi_B / dt$$

the integral is over an area "roofing" the path. A changing magnetic flux through the loop induces an emf.

Equations for Electricity and Magnetism

• Mangnetostatics:

$$\oint \vec{B} \cdot \vec{d\ell} = \mu_0 I$$

around any closed curve: *I* is the total current flow across any surface roofing the closed curve of integration.

- But is this the whole story?
- Fields changing in time changed the electrostatic equation, what about this magnetostatic equation?
- Let's look at a particular case...

Spherical Current

- At t = 0, a perfectly spherical ball of charge is placed at the center of a very large spherical conductor. The charge flows away equally in all directions. What is the magnetic field generated?
- 1) It points outwards equally in all directions
- 2) Same but pointing inwards
- 3) It circles around the initial sphere
- 4) No magnetic field is produced by these currents

Those Spherical Currents...

• Cannot produce a magnetic field!

- The configuration has perfect spherical symmetry—it would not be changed by turning it through an angle about any axis.
- The only fields satisfying this would point in or out along radii everywhere—but that could only happen with a net magnetic charge (N or S) at the center. So, no field at all...

Ampère's Law and Spherical Currents

- Imagine in 3D currents flowing spherically outward symmetrically from a ball of charge injected into a large conducting medium.
- Imagine a circular curve, like a crown, placed above the source. Clearly some of the current flows through a surface roofing this loop, so μ₀ is nonzero.
- But $\int \vec{B} \cdot d\vec{\ell} = 0$ around the loop, because the field \vec{B} is zero everywhere!
- So Ampère's law is not the whole story...

Another Ampère's Law Paradox

- Suppose now a capacitor is being charged by a steady current in a wire.
- Consider Ampère's law for a circular contour around the wire—it's supposed to be the same for any surface S roofing the circle, but we could choose S₂, going between the plates, so no current crosses it!



Maxwell's Solution



 Maxwell knew Faraday had generalized the electrostatic law to include a time-varying magnetic field by adding the changing flux through the curve:

$$\oint \vec{E} \cdot \vec{d\ell} = -d / dt \left(\int \vec{B} \cdot \vec{dA} \right) = -d\Phi_B / dt$$

 He noticed that when Ampère's law failed, there was a time-varying electric field through the surface roofing the curve, and suggested including it like this:

$$\oint \vec{B} \cdot \vec{d\ell} = \mu_0 \left(I + \frac{d}{dt} \left(\varepsilon_0 \int \vec{E} \cdot \vec{dA} \right) \right) = \mu_0 I + \mu_0 \varepsilon_0 d\Phi_E / dt$$

• (Writing $\int \vec{E} \cdot \vec{dA} = \Phi_E$, the electric flux.)

Why the Two Surfaces Give the Same Result

- The current *I* flowing through surface S₁ is the rate of change of charge on the top capacitor plate, *I* = dQ/dt.
- If the plates are close, all the electric field from the top plate will point down, none will cross S_1 , so $\int_{S_1+S_2} \vec{E} \cdot \vec{dA} = Q / \varepsilon_0$ and



$$\frac{d}{dt}\varepsilon_0 \int_{S_2} \vec{E} \cdot \vec{dA} = \varepsilon_0 \frac{d\Phi_E}{dt} = \frac{dQ}{dt} = I$$

Bottom line: the rate of change of electric flux through S_2 = current through S_1 .

Ampère's Law and Charge Conservation

- Ampère's law cannot work by itself for all surfaces spanning a circle like this:
- The surface S_2 is drawn to avoid the current. This is only possible because charge is piling up. The rate of change of electric flux just equals $1/\mathcal{E}_0$ times how fast the charge is piling up, from Gauss' law.



• This must equal the ingoing <u>current</u>—so the integral over S_1 = that over S_2 .

Maxwell's Equations

 The four equations that together give a complete description of electric and magnetic fields are known as Maxwell's equations:

$$\int \vec{E} \cdot \vec{dA} = q / \varepsilon_0 \qquad \int \vec{B} \cdot \vec{dA} = 0$$
$$\oint \vec{E} \cdot \vec{d\ell} = -d\Phi_B / dt$$
$$\oint \vec{B} \cdot \vec{d\ell} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Maxwell himself called this term the "displacement current": it produces magnetic field like a current.



Magnetic Field In Charging Capacitor

• Taking the field between plates uniform, use

$$\oint \vec{B} \cdot \vec{d\ell} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot \vec{dA}$$

- For a disc surface between the plates, there is no current *I* through the surface, there is a changing electric field uniform over the area, generating a circular magnetic field.
- For the *total* plate area, $\varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot \vec{dA} = \varepsilon_0 \frac{d\Phi_E}{dt} = \frac{dQ}{dt} = I$

Magnetic Field In Charging Capacitor

- For the total area $\varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot \vec{dA} = \frac{dQ}{dt} = I$
- For a disc of radius *r*, the total changing electric field is given by

$$\varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot \vec{dA} = I \frac{\pi r^2}{\pi R^2}$$

• Now use
$$\oint \vec{B} \cdot \vec{d\ell} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot \vec{dA}$$

to find
$$B = \frac{\mu_0 I}{2\pi} \frac{r}{R^2}$$
 between plates.

Charging Capacitor and Betatron

- Recall that in the betatron a uniform magnetic field increasing in strength in time generated a circling electric field that could be used to accelerate charged particles.
- In the charging capacitor we've been looking at, a uniform electric field increasing in strength in time generates a circling magnetic field.
- In regions of space where there are no charges, changing electric and magnetic fields are related to each other in a very symmetric way.

Clicker Question (Review)

- Suppose you have an infinite uniform plane of electric charge. What is its electric field?
- A. Parallel to the plane, of uniform strength.
- B. Parallel to the plane, decreasing strength with distance from the plane.
- C. Perpendicular to the plane, of uniform strength throughout space.
- D. Perpendicular to the plane, decreasing strength with distance from the plane.

Clicker Answer

• C: Perpendicular to the plane, of uniform strength throughout space.

 An infinite plane is of course an idealization: but for a uniform plane charge distribution of finite size, the electric field has very close to uniform strength for distances from the plane less than the linear size of the charge distribution.

Clicker Question

- Suppose now the uniformly charged plane is set in motion with constant velocity. This means we have a plane of electric current.
- The magnetic field generated by this current:
- A. Is perpendicular to the plane.
- B. Is parallel to the plane, and in the same direction as its velocity.
- C. Is parallel to the plane, and perpendicular to the velocity direction.

Clicker Answer

- Is parallel to the plane, and <u>perpendicular</u> to the velocity direction.
- Remember the Biot Savart law: $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{r^2}$
- The magnetic field from a small piece of current is perpendicular to the current direction—but in this moving plane, all current flow is in the same direction, so all fields are perpendicular to that direction.

Magnetostatic Field from a Sheet of Current (no net charge: current in metal sheet)

- A large uniform sheet of electric current: think of it as many parallel close wires perpendicular to the screen, current flowing downwards, *I* amps per meter.
- What is the magnetic field? There can be no perp field.
- It's OK to use Ampère's law with rectangular contour, the enclosed current is *IL*.



$$\oint \vec{B} \cdot \vec{d\ell} = IL \text{ gives } B = \mu_0 I/2.$$

Switching on the Current Sheet

- If the current sheet is suddenly switched on, in the first moments the magnetic field is only established close to the sheet.
- We'll assume it moves out like a tidal wave away from the sheet, at speed v, so at time t it extends out to vt, with nothing beyond.



(No *B*-field out here yet.)

Ampère's Law at Time t

- This rectangular contour still includes current *LI*, but clearly $\oint \vec{B} \cdot \vec{d\ell} = 0$.
- What's going on?
- We know the correct equation is really $\oint \vec{B} \cdot \vec{d\ell} = \mu_0 LI + \mu_0 \varepsilon_0 d\Phi_E / dt$
- This will be correct only if there is also an electric field <u>perpendicular to the loop</u>, its flux increasing with time.



(No **B**-field out here yet.)

Ampère's Law at Time t

- As soon as the expanding magnetic field reaches our loop, $\oint \vec{B} \cdot \vec{d\ell} = \mu_0 LI$ and there can be no further change in the perpendicular electric field.
- This means the electric field *E* is spreading right along with the magnetic field, at *v*, so



(No **B**-field out here yet.)

 $d\Phi_E / dt = 2EvL$ and from $0 = \oint \vec{B} \cdot \vec{d\ell} = \mu_0 LI + \mu_0 \varepsilon_0 d\Phi_E / dt$ we find for field strengths $B = \mu_0 I / 2 = \mu_0 \varepsilon_0 vE$

Picturing the Fields...

- The current sheet is in the xy-plane, current in the -x direction.
- At time *t* after switch on, the fields will have reached *vt* as shown (we show one way—fields go -*z* too).
- We haven't yet used $\oint \vec{E} \cdot \vec{d\ell} = -d / dt \left(\int \vec{B} \cdot \vec{dA} \right)$
- What does that tell us?



Picturing the Fields...

So let's look a

 $\oint \vec{E} \cdot \vec{d\ell} = -d / dt \left(\int \vec{B} \cdot \vec{dA} \right)$

- Take a rectangular contour, two sides parallel to the electric field, one side beyond v: the integral gives EL = vLB.
- (we're not worrying about sign—these are field amplitudes.)



The Speed of Light

• To summarize: for the outward traveling magnetic and electric fields from a switched-on current sheet, the equation $\oint \vec{B} \cdot \vec{d\ell} = \mu_0 L I + \mu_0 \varepsilon_0 d\Phi_E / dt$

gives $B = \mu_0 I / 2 = \mu_0 \varepsilon_0 v E$.

- The equation $\oint \vec{E} \cdot \vec{d\ell} = -d / dt \left(\int \vec{B} \cdot \vec{dA} \right)$ gives $\vec{E} = v\vec{B}$.
- They both give the ratio <u>B/E</u>— and that fixes v!

$$\frac{B}{E} = \mu_0 \varepsilon_0 v = \frac{1}{v}, \text{ so } v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m.sec}^{-1}$$

The Speed of Light

- This outgoing wave could have been made harmonic simply by oscillating the current in the sheet.
- The wave's outgoing speed is fully determined by μ_0 , which—remember—we defined as $4\pi \times 10^{-7}$, and by \mathcal{E}_0 , which is measured in electrostatic experiments.
- But the speed is <u>exactly</u> that of light!
- Maxwell concluded that light is an electromagnetic wave.

The Electromagnetic Spectrum

 The equations place no restriction on possible wavelengths of these electromagnetic waves. It follows that light, with wavelengths only between 400 nm and 750 nm, is a small part of a vast electromagnetic spectrum—see the next slide...

