Waves II

Physics 2415 Lecture 26

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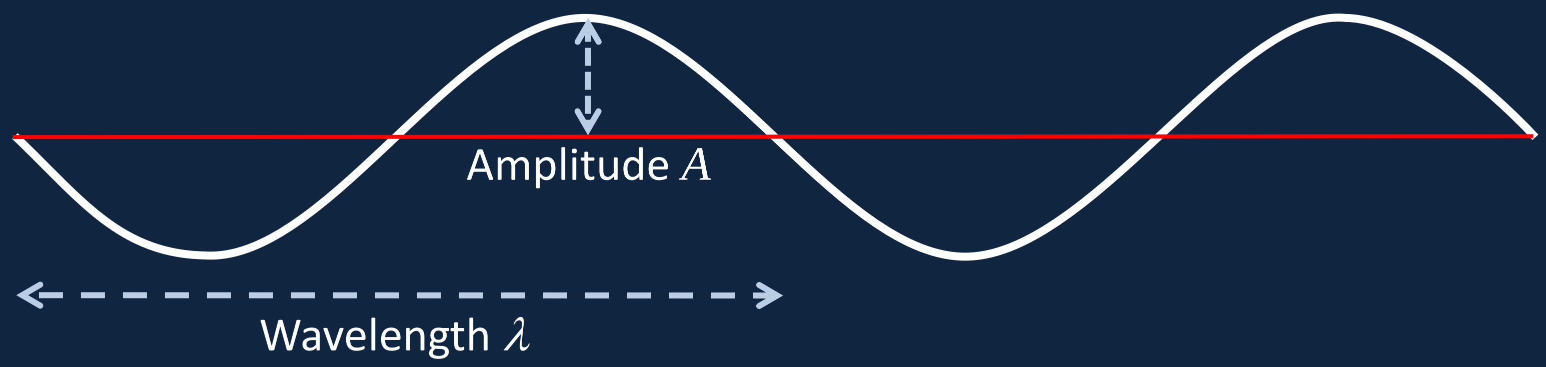
Today's Topics

- The wave equation
- Energy and power of waves
- Superposition
- Standing waves as sums of traveling waves
- Fourier series

Harmonic Waves

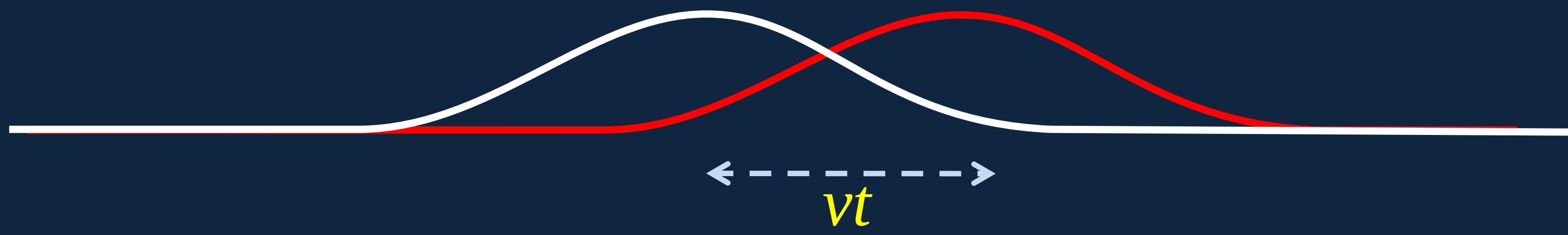
- A simple harmonic wave has sinusoidal form:



- For a string along the *x*-axis, this is local displacement in *y*-direction at some instant.
- For a sound wave traveling in the *x*-direction, this is local *x*-displacement at some instant.

Traveling Wave

- Experimentally, a pulse traveling down a string under tension maintains its shape:



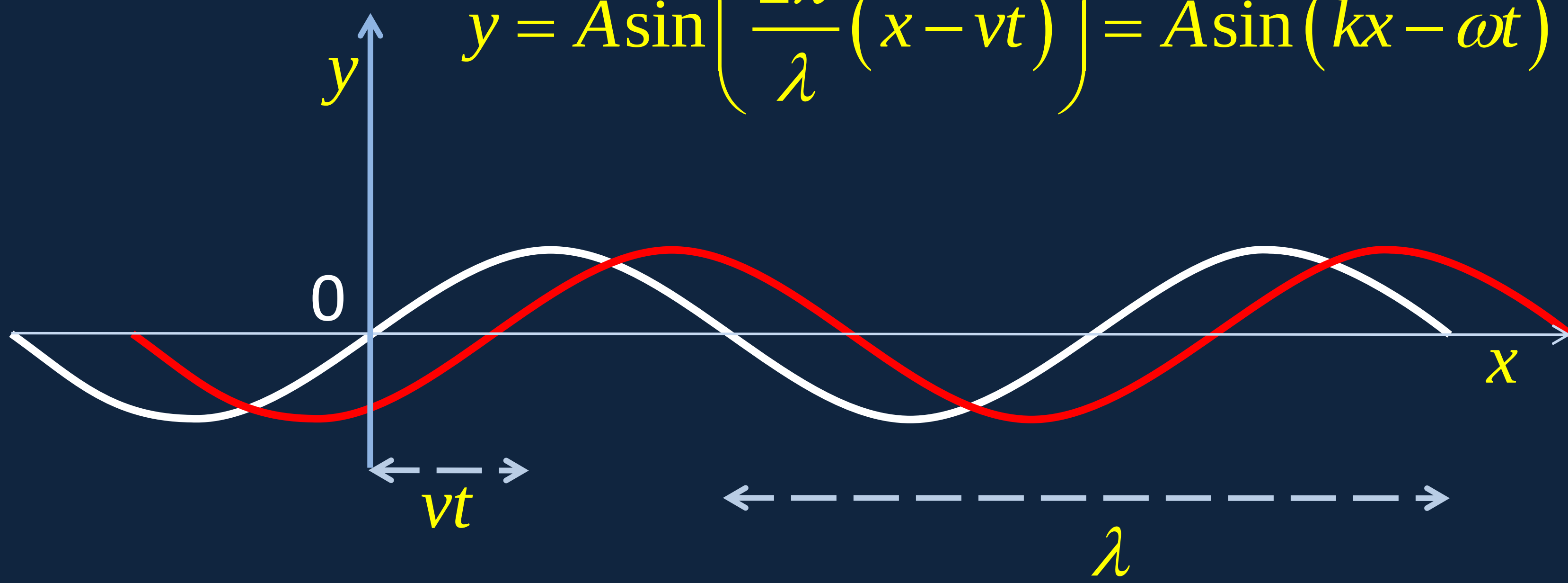
- Mathematically, this means the perpendicular displacement y stays the same function of x, but with an origin moving at velocity v:

$$y = f(x,t) = f(x - vt)$$

Traveling Harmonic Wave

- A sine wave of wavelength λ, amplitude A, traveling at velocity v has displacement

$$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right) = A\sin(kx - \omega t)$$



Harmonic Wave Notation

- A sine wave of wavelength λ, amplitude A, traveling at velocity v has displacement

$$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

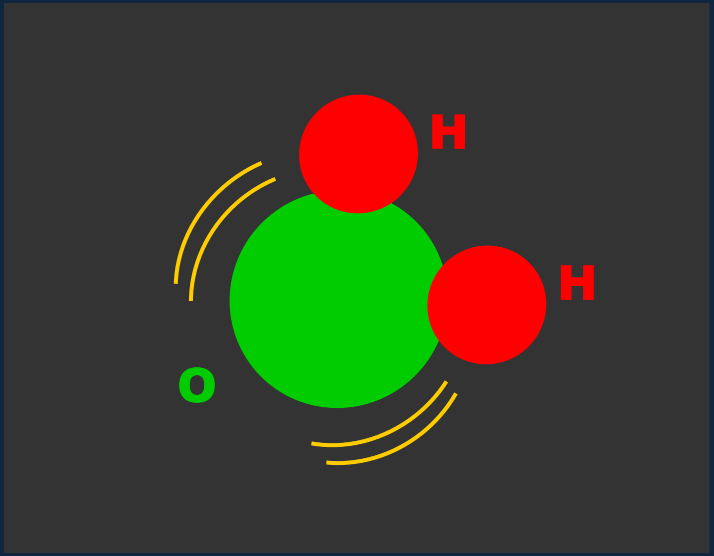
- This is usually written $y = A\sin(kx - \omega t)$, where the "wave number" $k = 2\pi/\lambda$ and $\omega = vk$.
- As the wave is passing, a single particle of string has simple harmonic motion with frequency ω radians/sec, or $f = \omega/2\pi$ Hz. Note that $v = \lambda f$

ConcepTest 15.5 Lunch Time

Microwaves travel with the speed of light, $c = 3 \times 10^8$ m/s. At a frequency of 10 GHz these waves cause the water molecules in your burrito to vibrate. What is their wavelength?

1 GHz = 1 Gigahertz = 10^9 cycles/sec

1) 0.3 mm
2) 3 cm
3) 30 cm
4) 300 m
5) 3 km



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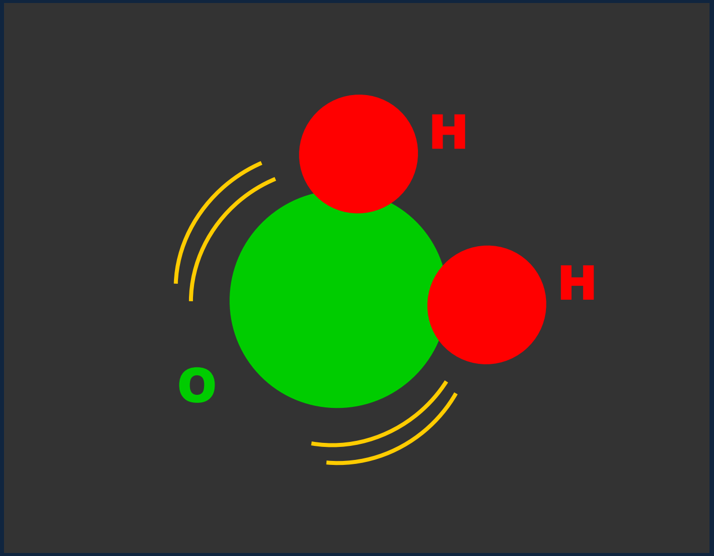
We know $v_{wave} = \frac{\lambda}{T} = f\lambda$

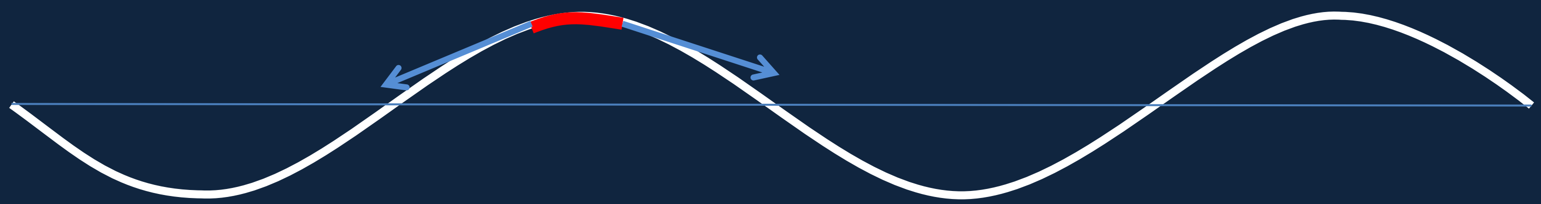
so $\lambda = \frac{v}{f} = \frac{3 \times 10^8 \text{ m/s}}{10 \times 10^9 \text{ Hz}}$

$\lambda = 3 \times 10^{-2}$ m $= 3$ cm



The Wave Equation

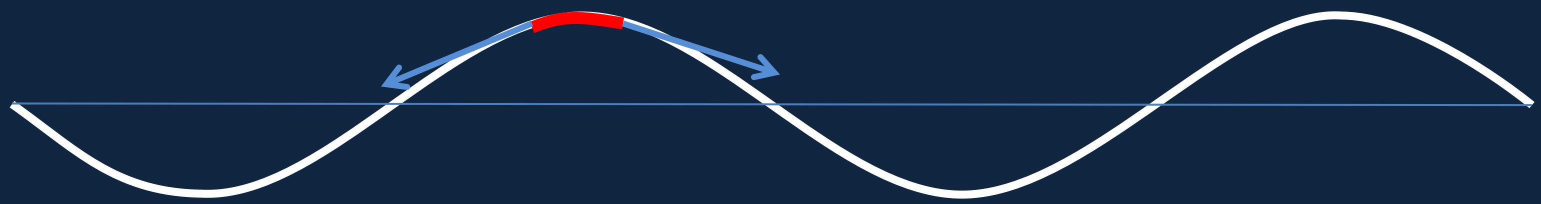
- The wave equation is just Newton's law $F = ma$ applied to a little bit of the vibrating string:



- The tiny length of string shown in red has length $m = \mu dx$, is accelerating in the y-direction with acceleration $a = \partial^2 f(x,t) / \partial t^2$, and the force F is the sum of the tensions at the two ends of the bit of string, which don't cancel because they're not parallel. Animation!

The Wave Equation

- The y-direction component of the tension T at the front end of the string is just T multiplied by the slope (for small amplitudes), $T\partial f\left(x+dx,t\right)/\partial x$.
- At the back end, T points backwards, so the downward component is $-T\partial f\left(x,t\right)/\partial x$.



- The total y-direction force is therefore

$$F=T\partial f\left(x+dx,t\right)/\partial x-T\partial f\left(x,t\right)/\partial x=T\left(\partial^2 f\left(x,t\right)/\partial x^2\right)dx$$

Wave Equation

- We're ready to write $F = ma$ for that bit of string:

$$F = T\partial f\left(x+dx,t\right)/\partial x - T\partial f\left(x,t\right)/\partial x = T\left(\partial^2 f\left(x,t\right)/\partial x^2\right)dx$$

- $m = \mu dx,\ a = \partial^2 f\left(x,t\right)/\partial t^2$.
- Putting it all together:

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}}$$

- Since this is nothing but Newton's second law, it must be true for *any* wave on a string.

Traveling Wave Equation

- Recall that from observation a traveling wave has the form $y = f(x - vt)$.
- From the chain rule, for that function

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial (x - vt)} \frac{\partial (x - vt)}{\partial t} = -v \frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

- Comparing this with the wave equation, we see that

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}}$$

This <u>proves</u> that $v = \sqrt{T / \mu}$.

ConcepTest 15.6a Wave Speed I

A wave pulse can be sent down a rope by jerking sharply on the free end. If the tension of the rope is increased, how will that affect the speed of the wave?

1) speed increases
2) speed does not change
3) speed decreases

ConcepTest 15.6a Wave Speed I

A wave pulse can be sent down a rope by jerking sharply on the free end. If the tension of the rope is increased, how will that affect the speed of the wave?

1) speed increases
2) speed does not change
3) speed decreases

The wave speed depends on the square root of the tension, so if the tension increases, then the wave speed will also increase.

ConcepTest 15.6b Wave Speed II

A wave pulse is sent down a rope of a certain thickness and a certain tension. A second rope made of the same material is twice as thick, but is held at the same tension. How will the wave speed in the second rope compare to that of the first?

1) speed increases
2) speed does not change
3) speed decreases

ConcepTest 15.6b Wave Speed II

A wave pulse is sent down a rope of a certain thickness and a certain tension. A second rope made of the same material is twice as thick, but is held at the same tension. How will the wave speed in the second rope compare to that of the first?

1) speed increases
2) speed does not change
3) speed decreases

The wave speed goes inversely as the square root of the mass per unit length, which is a measure of the inertia of the rope. So in a thicker (more massive) rope at the same tension, the wave speed will decrease.

ConcepTest 15.6c Wave Speed III

A length of rope *L* and mass *M* hangs from a ceiling. If the bottom of the rope is jerked sharply, a wave pulse will travel up the rope. As the wave travels upward, what happens to its speed? Keep in mind that the rope is ***not*** massless.

1) speed increases
2) speed does not change
3) speed decreases

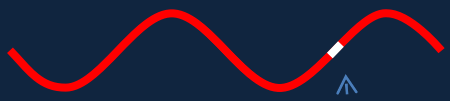
ConcepTest 15.6c Wave Speed III

A length of rope *L* and mass *M* hangs from a ceiling. If the bottom of the rope is jerked sharply, a wave pulse will travel up the rope. As the wave travels upward, what happens to its speed? Keep in mind that the rope is ***not*** massless.

1) speed increases
2) speed does not change
3) speed decreases

The tension in the rope is not constant in the case of a massive rope! The tension increases as you move up higher along the rope, because that part of the rope has to support all of the mass below it! Because the tension increases as you go up, so does the wave speed.

Harmonic Wave Energy



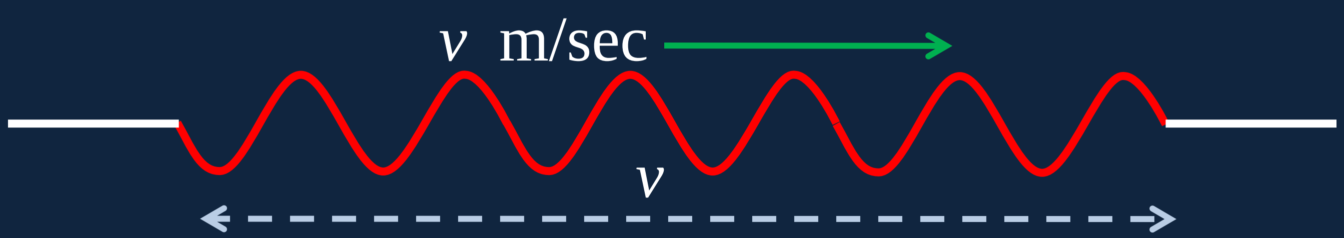
- Writing the wave $y = A\sin(kx - \omega t)$ where remember $k = 2\pi/\lambda, \quad \omega = vk$ it's clear that at any fixed point x a bit of string dx is oscillating up and down in simple harmonic motion with amplitude A and frequency $f = \omega/2\pi$ Hz.
- The energy of that bit dx is <u>all</u> kinetic when $y = 0$, ($kx = \omega t$), the y-velocity at that instant is

$$v = \partial y/\partial t = -\omega A\cos(kx - \omega t) = -\omega A$$

so the <u>total</u> energy in dx is $\frac{1}{2}mv^2 = \frac{1}{2}(\mu dx)A^2\omega^2$.

Harmonic Wave Energy

- The <u>total</u> energy in dx is $\frac{1}{2}mv^2 = \frac{1}{2}(\mu dx)A^2\omega^2$, so in length L the wave energy is $\frac{1}{2}\mu L A^2\omega^2$.
- Imagine now a group of waves, choose length *v*, moving to the right at speed *v* (passes you in just one second!):



- The power delivered by the waves is the energy passing a fixed point per second—that is

$$\bar{P} = \frac{1}{2}\mu v A^2\omega^2 = 2\pi^2\mu v A^2 f^2$$

The Wave Equation and Superposition

- If you have two solutions to the wave equation, $y = f(x,t)$ and $y = g(x,t)$, then $y = f + g$ is also a solution to the wave equation!
- This can be checked with the actual equation:

$$\frac{\partial^2 (f+g)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 (f+g)}{\partial t^2}$$

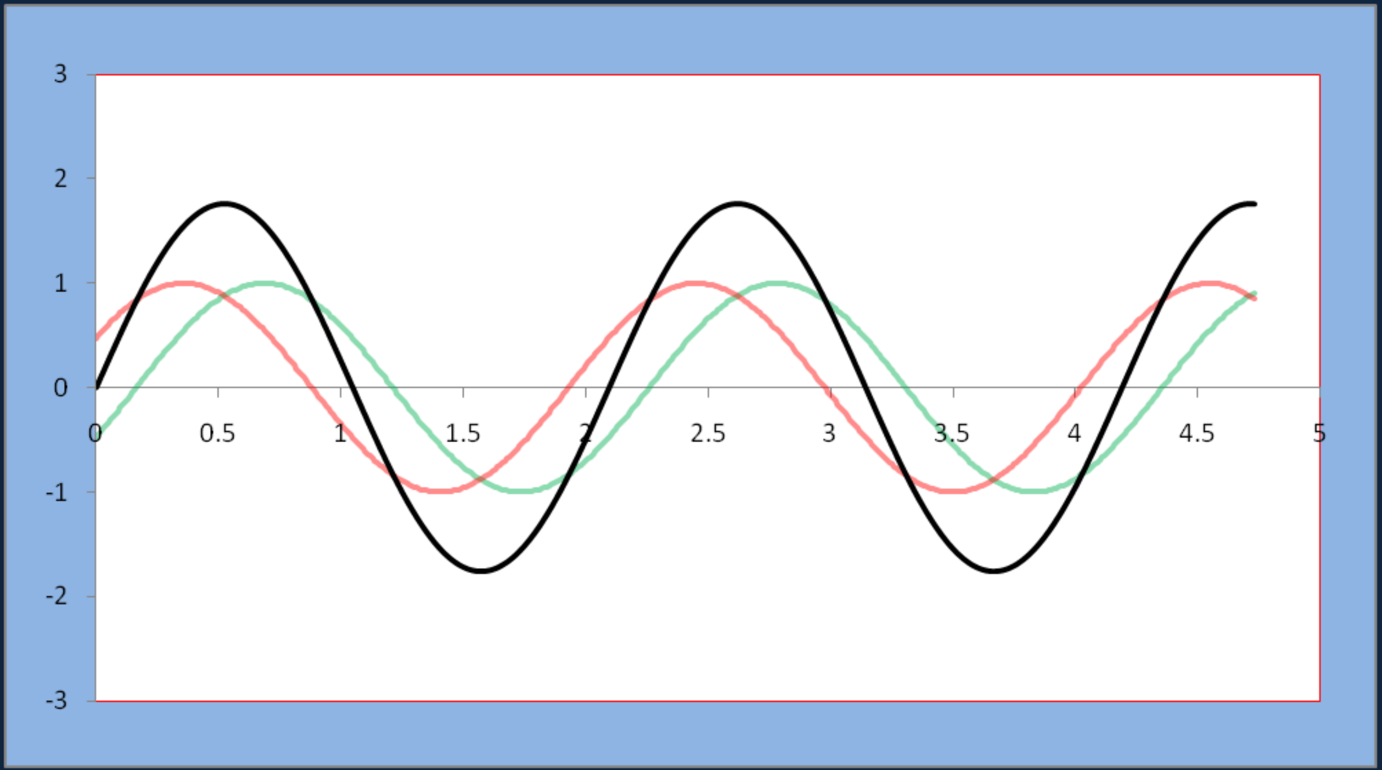
- Differential equations with this property are called “linear”. It means you can build up any shape wave from harmonic waves.

A Harmonic Wave Hits a Wall…

- When a wave hits a wall, the energy and wave form are reflected, and must be added to the incoming wave.
- What does this look like? Let's take the case of a wave on a string, the string fixed at one end…
- **Here it is…**

Harmonic Wave Addition

Two harmonic waves of the same wavelength and amplitude, but moving in opposite directions, add to give a standing wave.



Notice the standing wave also satisfies $\lambda f = v$, even though it's not traveling!

Standing Wave Formula

- To add two traveling waves of equal amplitude and wavelength moving in opposite directions, we use the trig formula for addition of sines:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

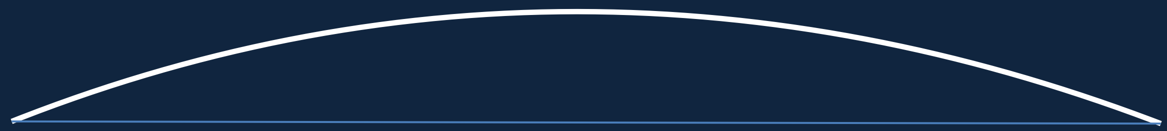
- Applying this,

$$A\sin\left(kx-\omega t\right)+A\sin\left(kx+\omega t\right)=2A\sin kx\cos\omega t$$

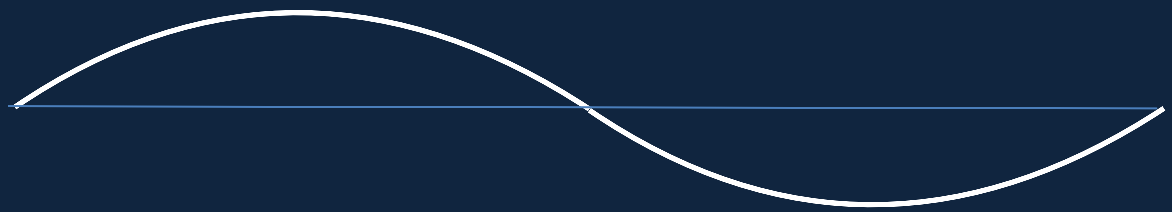
- Allowed values of k are given by $k\ell = \pi, 2\pi, 3\pi \ldots$ where ℓ is the string length.

Harmonic Wave on String

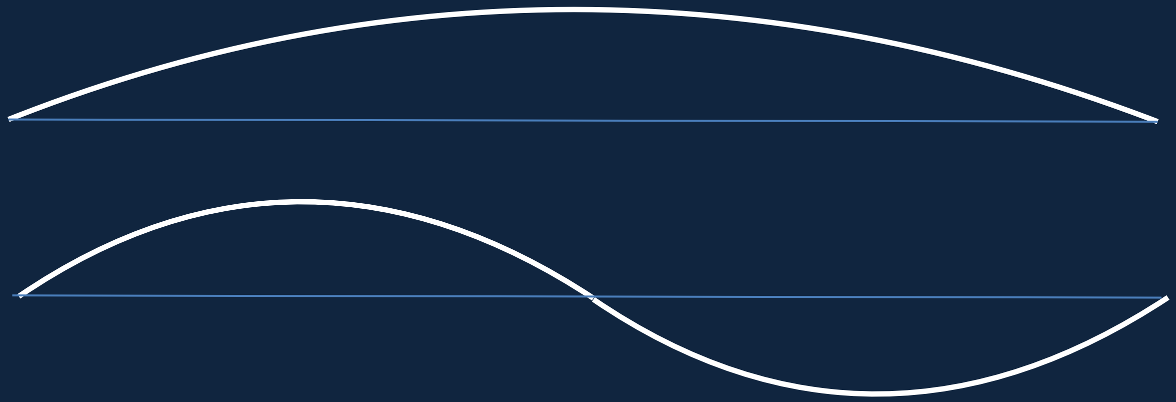
- The amplitude must always be zero at the ends of the string. From $\lambda v = f$, the lowest frequency note (the fundamental, or first harmonic) has the longest allowed wavelength: $\lambda = 2\ell$.



- The second harmonic has $\lambda = \ell$:

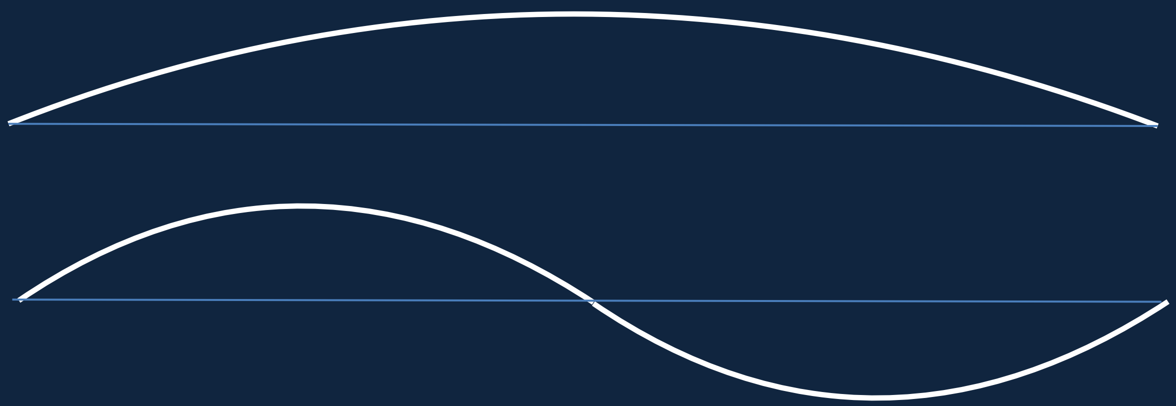


Clicker Question



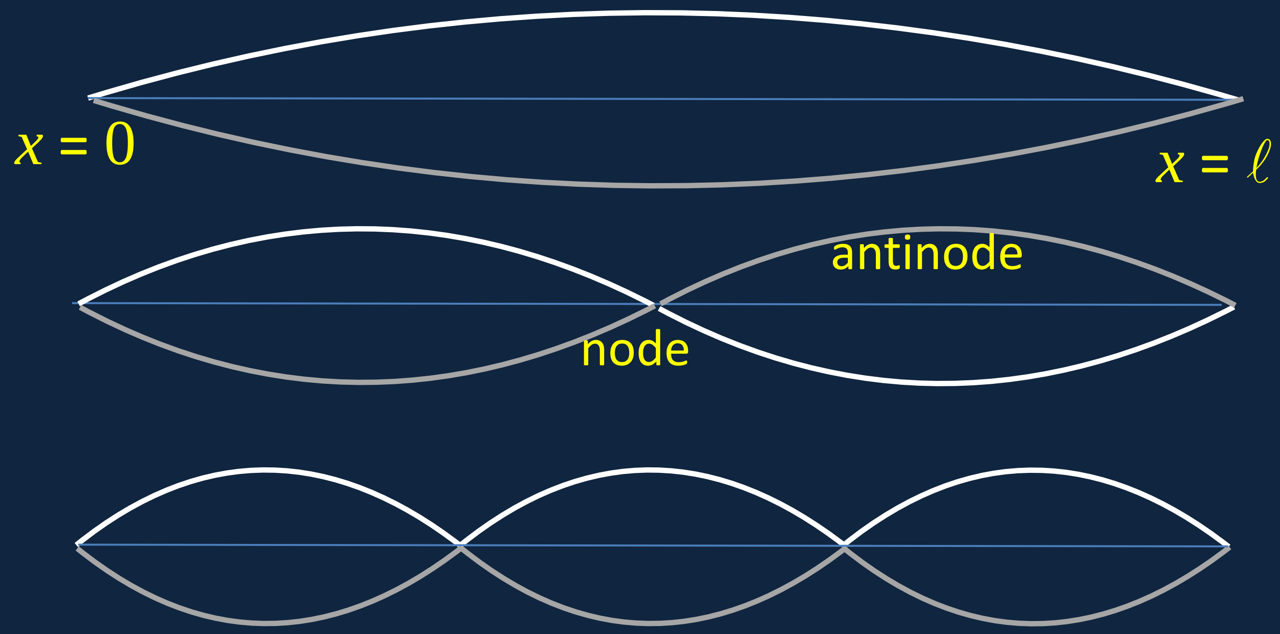
- For standing waves of equal amplitude on identical strings at the same tension, one string vibrating in the first harmonic mode, the other the second harmonic, the energy in the second harmonic string is:
- A. twice that in the first harmonic string
- B. four times…
- C. equal to…

Clicker Answer



- For standing waves of equal amplitude on identical strings at the same tension, one string vibrating in the first harmonic mode, the other the second harmonic, the energy in the second harmonic string is:
- A. twice that in the first harmonic string
- B. four times… ← $E = \frac{1}{2}\mu L A^2 \omega^2$
- C. equal to…

Nodes and Antinodes



The standing wave has form $y\left(x,t\right)=A\sin kx\cos\omega t=A\sin\frac{2\pi x}{\lambda}\cos 2\pi ft$

For a pure note on a string with fixed ends, $\lambda = 2\ell, \ell, \frac{2}{3}\ell, \ldots$

At a node, the string never moves: $\sin\frac{2\pi x}{\lambda}=0, \quad x=0,\frac{1}{2}\lambda,\lambda,\frac{3}{2}\lambda,\ldots$

Clicker Question

- The tension in a guitar string of fixed length is increased by 10%. How does that change the wavelength of the second harmonic?
- A. It increases by 10%
- B. It increases by about 5%
- C. It decreases by 10%
- D. it decreases by about 5%
- It stays the same.

Clicker Answer

- The tension in a guitar string of fixed length is increased by 10%. How does that change the wavelength of the second harmonic?
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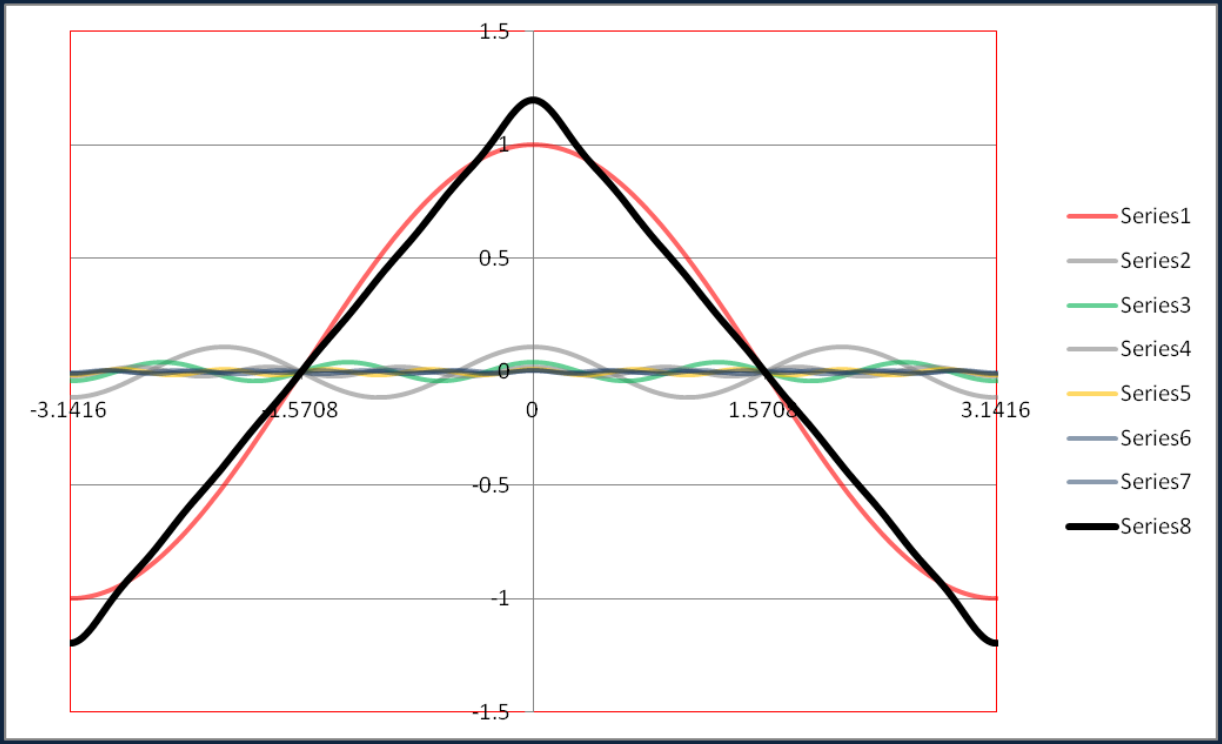
- B. It increases by about 5%
- C. It decreases by 10%
- D. it decreases by about 5%
- It stays the same: it’s just the length of the string!

Fourier Series

We can also build up any type of periodic wave by adding harmonic waves with the right amplitudes—this is called “Fourier analysis”: in music, it’s building up a complex note from its harmonics: here’s a triangle (formed by pulling an instrument string up at the midpoint then letting go?).



Pulse Encounter

It's worth seeing how two pulses traveling in opposite directions pass each other: