

## Describing Motion in Two and Three Dimensions: Vectors

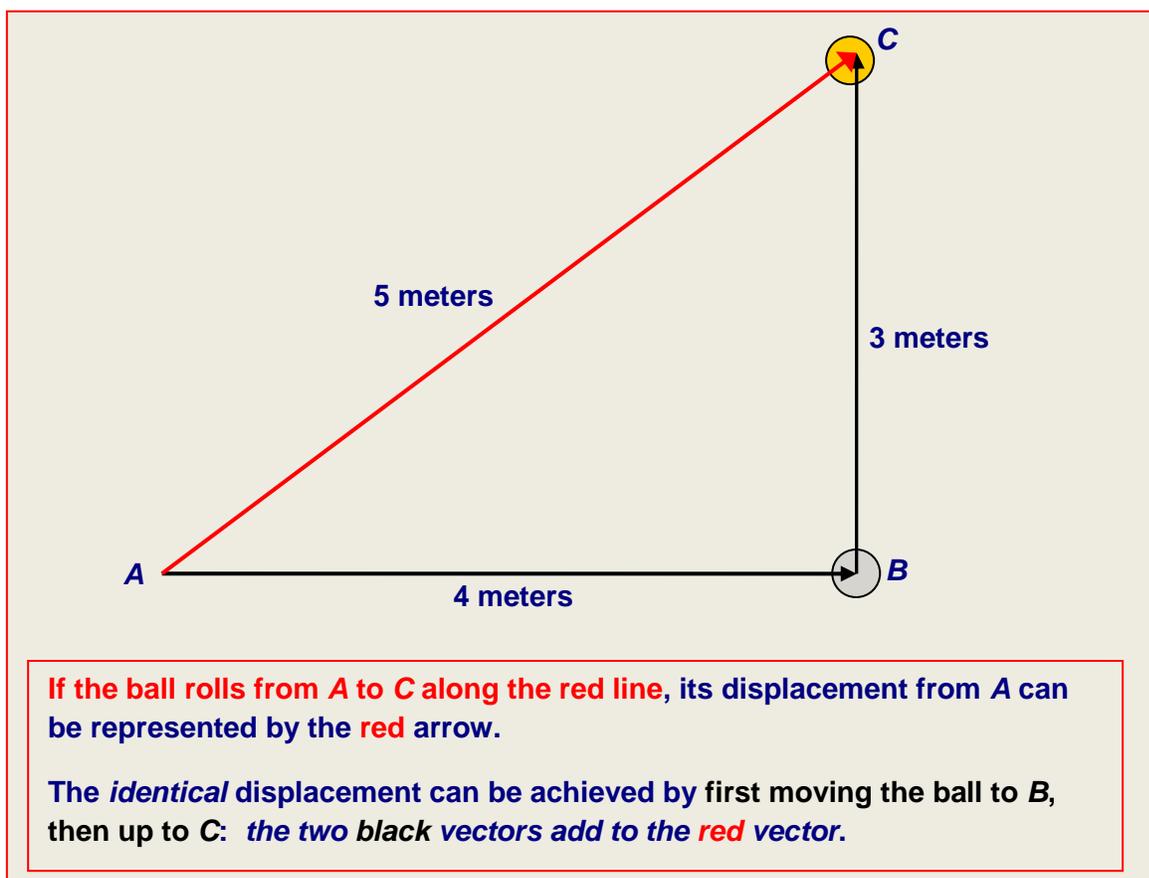
Michael Fowler, Physics 142E Lec 4. 20 Jan 2009

### Displacements

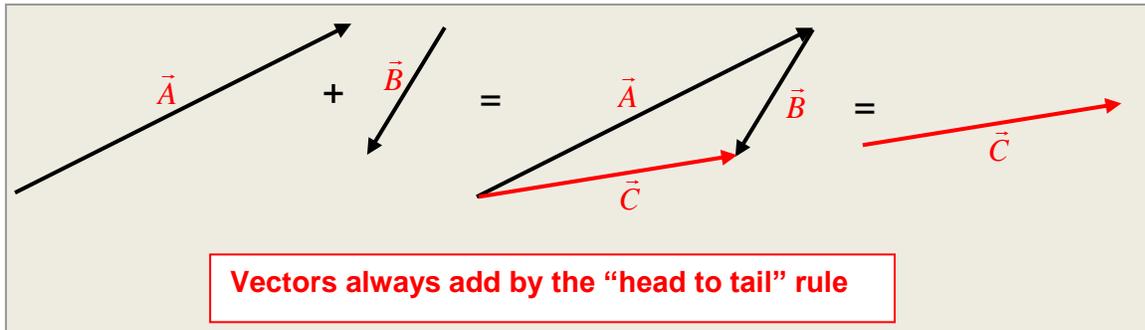
We've analyzed motion of an object (like a small ball) in one dimension fairly thoroughly, using the concepts of displacement (meaning position, or  $x$ -coordinate), velocity (rate of change of  $x$ -coordinate) and acceleration (rate of change of velocity). Let's now generalize these ideas to a ball being moved around—displaced—in a plane. For one dimension, displacement had a **magnitude** and a **sign** (forwards or backwards from the origin?).

In **two dimensions** displacement has a **magnitude** and a **direction**. A displacement of one meter from the origin could land you anywhere on a circle. The standard way of representing a displacement in a plane is with a **vector**, an arrow whose length indicates distance, with some agreed scale, such as 1 cm representing 1 m, and whose direction represents the direction of the displacement.

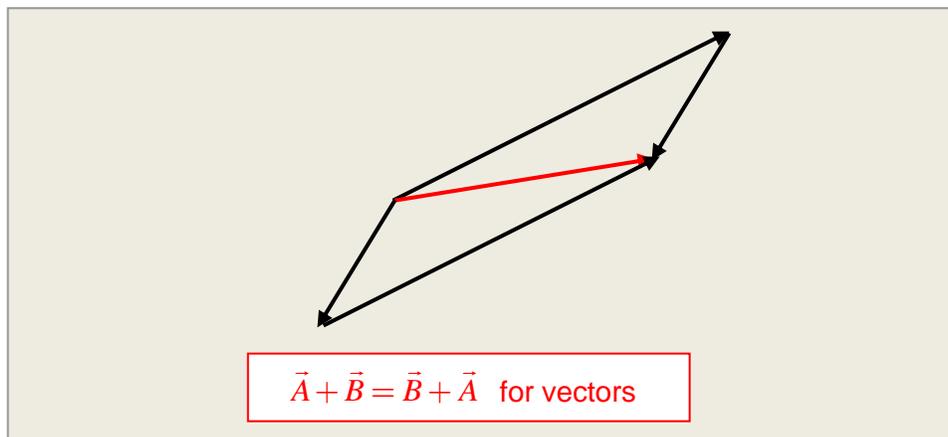
An equivalent representation is to give the displacement from the origin as a sum of displacements in the  $x$ -direction and the  $y$ -direction; these are called the **components** of the displacement.



In fact, this gives us the clue for adding vectors: it's just the way we add displacements, if I walk 5 steps in one direction, and represent that by a vector, then walk 2 steps in a different direction, another vector, evidently my total displacement from my starting position is given by adding the two displacements, putting the vectors "head to tail"



This addition is like making a map of the path taken—paths add up this way, so vector displacements do too. And, it doesn't matter which move you make first:



The magnitude (or length) of the vector  $\vec{A}$  is sometimes called the modulus, and can be denoted by  $|\vec{A}|$  or sometimes just  $A$ .

It's evident that if  $\vec{A}$  is a vector,  $\vec{A} + \vec{A}$  is a vector twice as long in the same direction—and we define the product of a vector  $\vec{A}$  and a positive number  $b$ ,  $b\vec{A}$ , to be a vector in the same direction as  $\vec{A}$ , of length  $bA$ . If  $b$  is a negative number, the vector  $b\vec{A}$  points in the opposite direction, so, for example,  $\vec{A} + (-\vec{A}) = 0$ .

It's also clear from the above **how to subtract vectors**: since  $\vec{A} + \vec{B} = \vec{C}$ , we have  $\vec{C} - \vec{A} = \vec{B}$ . From the first diagram above, we can see from this that the way to find the difference of two vectors, in this

case  $\vec{C} - \vec{A}$ , is to **put their tails together** and draw a vector  $\vec{B}$  **from the head of  $\vec{A}$  to the head of  $\vec{C}$** : **this vector is the difference vector** (check by adding it to  $\vec{A}$  to get  $\vec{C}$  !)

### Vector Components and Unit Vectors

As mentioned at the beginning of this section, any displacement from the origin can be written as a sum of displacements in the x-direction and the y-direction; these being called the **components** of the displacement.

More generally, it's clear that any two-dimensional vector is a sum of a vector in the x-direction and one in the y-direction, and in fact any three-dimensional vector has an additional contribution in the z-direction.

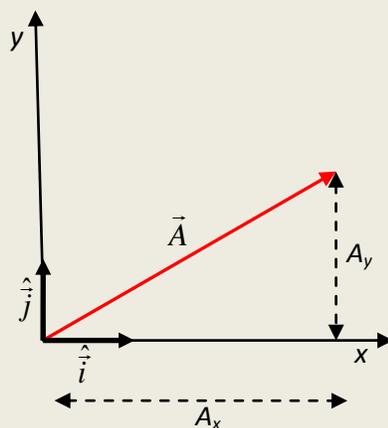
This decomposition is best represented by introducing three unit vectors (length 1) denoted  $\hat{i}$ ,  $\hat{j}$ ,  $\hat{k}$ , in the directions of the three axes, and writing a vector  $\vec{A}$  as:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}.$$

In fact, the vector is often written simply:

$$\vec{A} = (A_x, A_y, A_z).$$

The numbers  $A_x, A_y, A_z$  are called the **components** of the vector.



Vector, unit vectors and vector components in two dimensions

Note that from Pythagoras' theorem the length of the vector is related to the length of the components:  $A^2 = A_x^2 + A_y^2$ , and in three dimensions this becomes  $A^2 = A_x^2 + A_y^2 + A_z^2$ .

It's also worth noting that if the angle between  $\vec{A}$  and the x-axis is  $\vartheta$ , then  $A_x = A \cos \theta$ ,  $A_y = A \sin \theta$ .

The component representation is a handy alternative to the graphical approach: two vectors add together simply by adding their components, etc., for example:

$$\vec{A} - 2\vec{B} = (A_x - 2B_x, A_y - 2B_y, A_z - 2B_z).$$

All problems about vectors can be handled using components, but the graphical approach will often give more insight into what's going on.

## Vector Kinematics: Velocity and Acceleration Vectors

Recall that in lecture 2 we defined velocity along a line at a given time by

$$v(t) = \lim_{\substack{|t_2 - t_1| \rightarrow 0 \\ t_1 < t < t_2}} \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{dx}{dt}$$

The obvious generalization to three dimensions is:

$$\vec{v}(t) = \lim_{\substack{|t_2 - t_1| \rightarrow 0 \\ t_1 < t < t_2}} \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} = \frac{d\vec{r}}{dt}$$

where  $\vec{r}$  is the position vector.

Suppose now a particle is moving along a curved path in a plane. Then the direction of the velocity vector at some time is the direction of the vector  $\vec{r}_2 - \vec{r}_1$ , where these are two points very close together on the curve—if you draw the curve, you'll see this direction is just the tangent to the curve. Of course, as the particle moves along the curve, at any instant its velocity is in the direction of the curve.

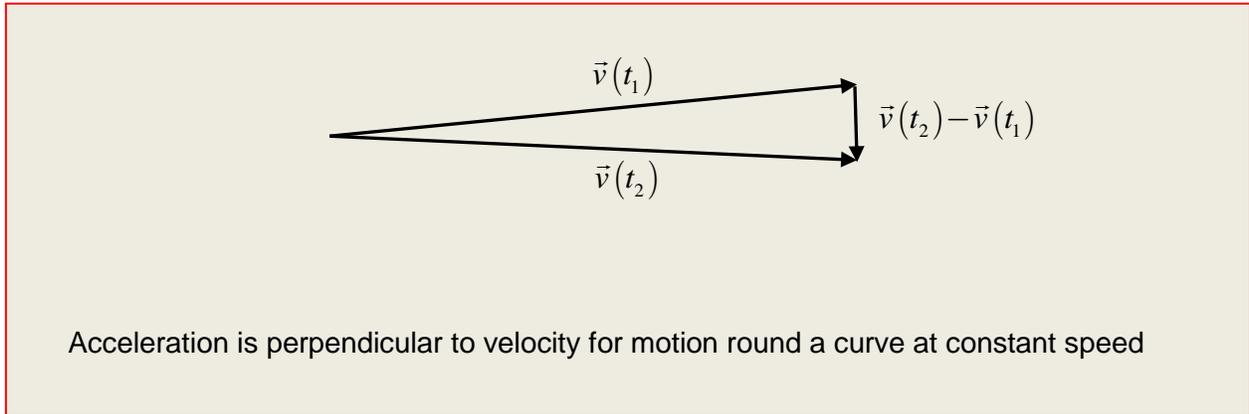
The direction of acceleration for motion in a plane is a lot less obvious. Generalizing the formulas for one dimension in lecture 2, average acceleration over a time period

$$\vec{a}(t) = \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1},$$

and acceleration at a given time  $t$

$$\vec{a}(t) = \lim_{\substack{|t_2 - t_1| \rightarrow 0 \\ t_1 < t < t_2}} \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1} = \frac{d\vec{v}}{dt}.$$

Formally, this is just like one dimension, but there's a new wrinkle—the **velocity can change without the speed changing**. The simplest example is a car going round a bend at constant speed: the velocity changes **direction**, so there's a nonzero change in velocity:

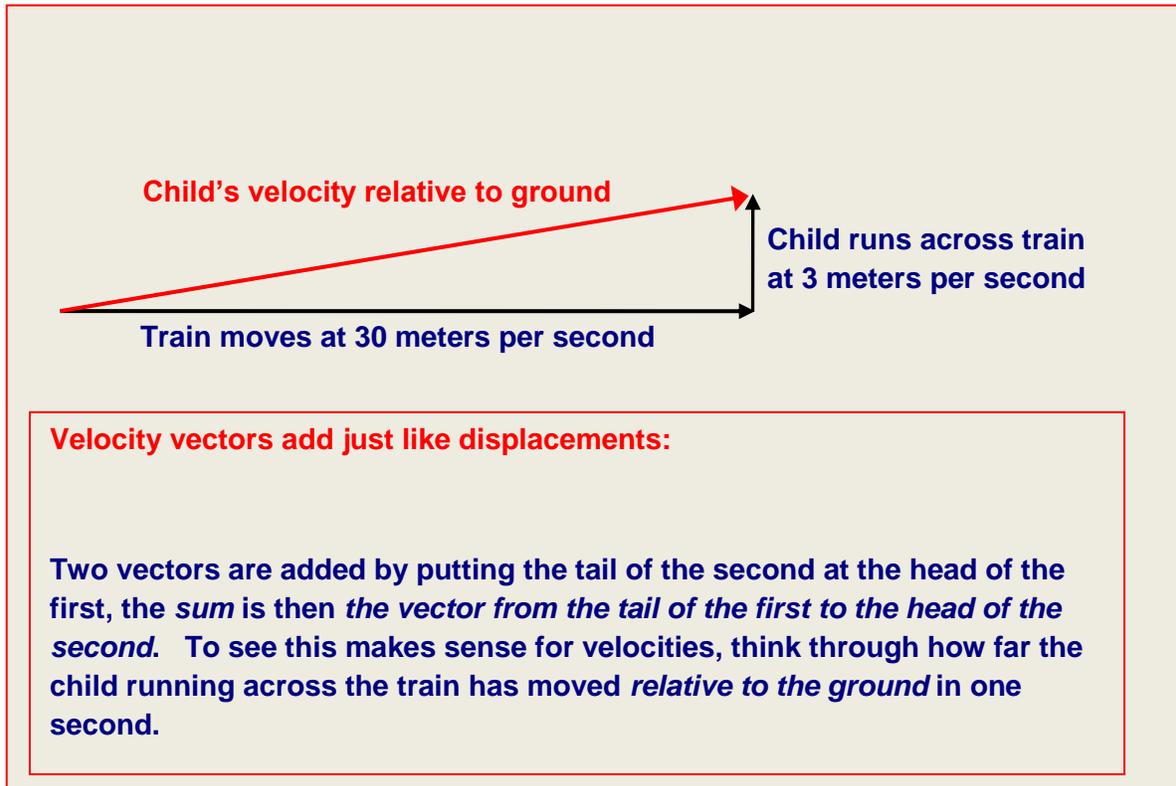


When we get to Newton's laws, we'll find that this not-so-obvious kind of acceleration is just as important as old-fashioned step-on-the-gas acceleration—both are the response of an object to an external force.

### Relative Velocities: a Child Running in a Train

Consider a child running at 3 meters per second (about 6 mph) in a train. The child is running parallel to the length of the train, towards the front, and the train is moving down the track at 30 meters per second. What is the child's velocity relative to the ground? It is 33 meters per second in the direction the train is moving along the track (notice we always specify *direction* for a velocity). To really nail this down, you should think through just how far the child moves relative to the ground in one second—three meters closer to the front of the train, and the train has covered 30 meters of ground.

A trickier point arises if the child is running *across* the train, from one side to the other. (This run will only last about one second!) Again, the way to find the child's velocity relative to the ground is to visualize how much ground the child covers in one second—three meters in the direction across the track, from one side to the other, plus thirty meters in the direction along the track.



To find the total velocity, we now have to add two velocities at right angles, using the “head to tail” rule for adding vectors. This is just the same problem as the ball rolling across the table at an angle discussed above, and we need to use Pythagoras’ theorem to find the child’s *speed* relative to the ground.

Similar examples include a swimmer’s velocity relative to the flowing river and to the river bank, a ferryboat that must aim upstream to get directly across a river, and an airplane crossing the jet stream. Of course, the velocity of *A* relative to *B*, and that of *B* relative to *C*, need not be at right angles, but the rules for vector addition apply in finding the velocity of *A* relative to *C*.