Damped Harmonic Motion

• In the real world, oscillators experience damping forces: friction, air resistance, etc.
• These forces always oppose the motion: as an example, we consider a force $F = -bv$ proportional to velocity.
• Then $F = ma$ becomes:

$$ma = -kx - bv$$

• That is, $md^2x / dt^2 + bdx / dt + kx = 0$
Underdamped Motion

- The equation of motion

\[ m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \]

has solution

\[ x = Ae^{-\gamma t} \cos \omega't \]

where

\[ \gamma = \frac{b}{2m}, \]
\[ \omega' = \sqrt{\left(\frac{k}{m}\right) - \left(\frac{b^2}{4m^2}\right)} \]

Plot: \( m = 1, k = 4, b = 0.11 \)
Underdamped Motion

- The point to note here is that the damping can cause rapid decay of the oscillations without a perceptible change in the period (around 0.04% for $b = 0.11$, $k = 4$, $m = 1$).
Underdamped Motion

• Compare the curve with the equation: the successive position maxima follow an exponential curve $Ae^{-\gamma t}$, so any maximum reached is, say, 90% of the previous maximum.

• Remember the energy at maximum displacement is $\frac{1}{2}kx^2$.

\[
x = Ae^{-\gamma t} \cos \omega' t
\]
Clicker Question

• The amplitude in a damped oscillator reaches half its original value after four cycles. At which point does the oscillator have only half its original energy?
  A. 2 cycles
  B. 4 cycles
  C. 8 cycles
Not So Underdamped Motion

Even when the damping absorbs 98% of the energy in one period, the change in the length of the period is only around 10%!
Critical Damping

- As the damping is further increased, the period lengthens until at $b^2 = 4mk$ it becomes infinite, and the amplitude decays exponentially.

- (Actually, in this one case a prefactor $A + Bt$ is needed to match initial conditions—we’ll ignore this minor refinement.)
Overdamping

- Doubling the damping beyond critical damping just doubles the time for the amplitude to decay by a given amount.
Ideal Damping for Shock Absorbers?

• Critical damping is not the best choice: underdamping gives a quicker response, and the overshoot can be very small.

• Explore this for yourself with this applet!
The Damped Driven Oscillator

• We now consider a damped oscillator with an external harmonic driving force.

• We’ll look at the case where the oscillator is well underdamped, and so will oscillate naturally at \( \omega_0 = \sqrt{k / m} \).

• The external driving force is in general at a different frequency, the equation of motion is:

\[
md^2x / dt^2 + bdx / dt + kx = F_0 \cos \omega t
\]
The Damped Driven Oscillator

- If the driving frequency is far from the natural frequency, there is only a small response, even with no damping. Here the driving frequency is about twice the natural frequency.
The Damped Driven Oscillator

- This shows the oscillator with the same strength of external driving force, but at its natural frequency.
- The amplitude increases until damping energy losses equal external power input: this is **resonance**.
- Applet link!
- Tacoma Narrows Bridge.