

Simple Harmonic Motion

Physics 1425 Lecture 28

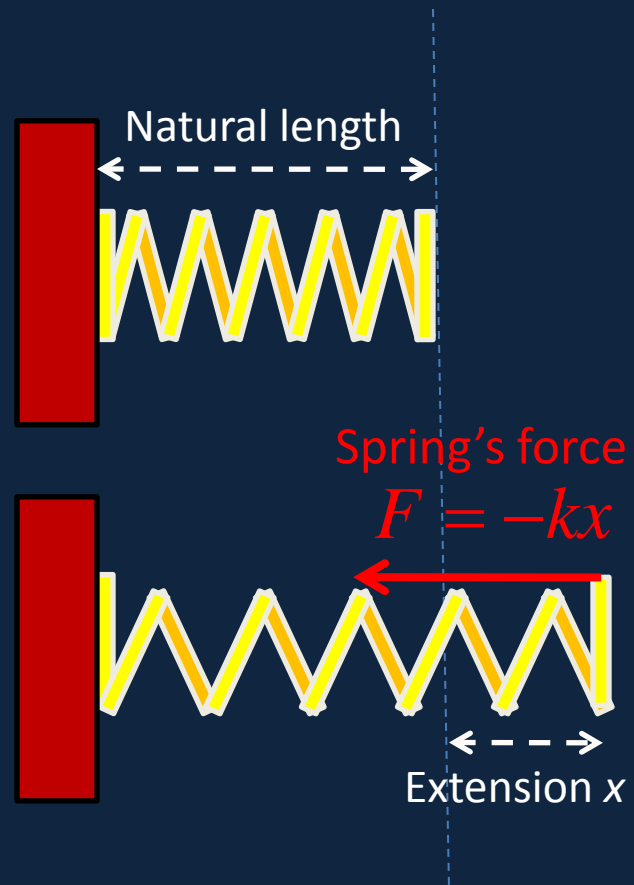
Force of a Stretched Spring

- If a spring is pulled to extend beyond its natural length by a distance x , it will pull back with a force

$$F = -kx$$

where k is called the “spring constant”.

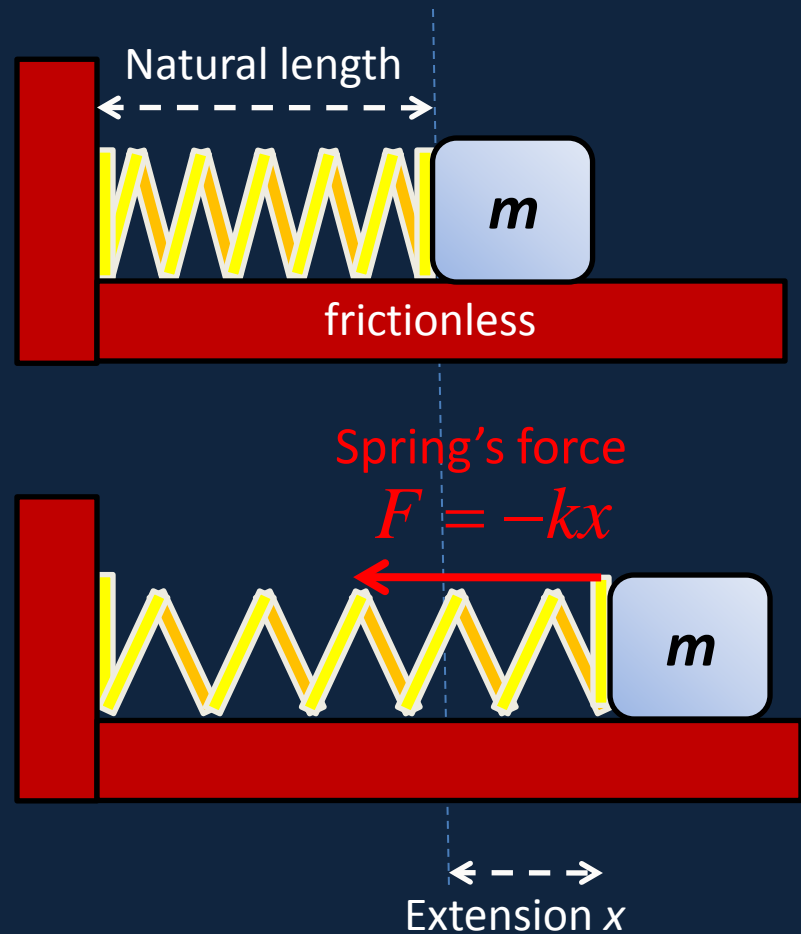
The same linear force is also generated when the spring is *compressed*.



Mass on a Spring

- Suppose we attach a mass m to the spring, free to slide backwards and forwards on the frictionless surface, then pull it out to x and let go.
- $F = ma$ is:

$$md^2x / dt^2 = -kx$$



Solving the Equation of Motion

- For a mass oscillating on the end of a spring,

$$m d^2 x / dt^2 = -kx$$

- The most general solution is

$$x = A \cos(\omega t + \phi)$$

- Here A is the amplitude, ϕ is the phase, and by putting this x in the equation, $m\omega^2 = k$, or

$$\omega = \sqrt{k / m}$$

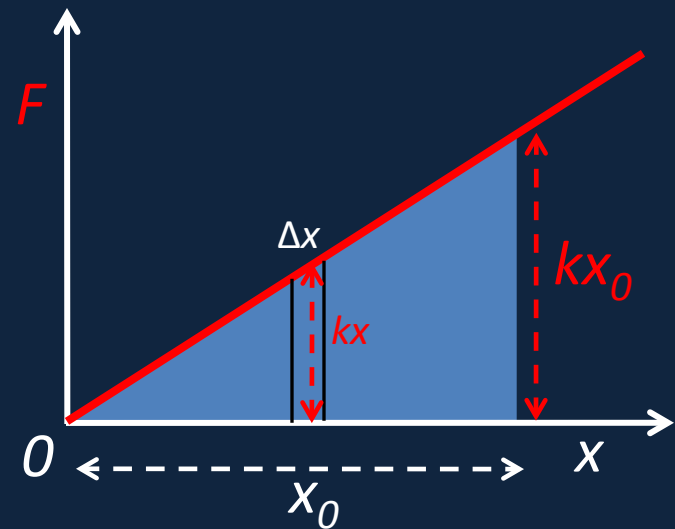
- Just as for circular motion, the time for a complete cycle

$$T = 1 / f = 2\pi / \omega = 2\pi \sqrt{m / k} \quad (f \text{ in Hz.})$$

Energy in SHM: Potential Energy Stored in the Spring

- Plotting a graph of external force $F = kx$ as a function of x , the work to stretch the spring from x to $x + \Delta x$ is force \times distance
- $\Delta W = kx\Delta x$, so the **total work to stretch the spring to x_0** is

$$W = \int_0^{x_0} kx dx = \frac{1}{2} kx_0^2$$



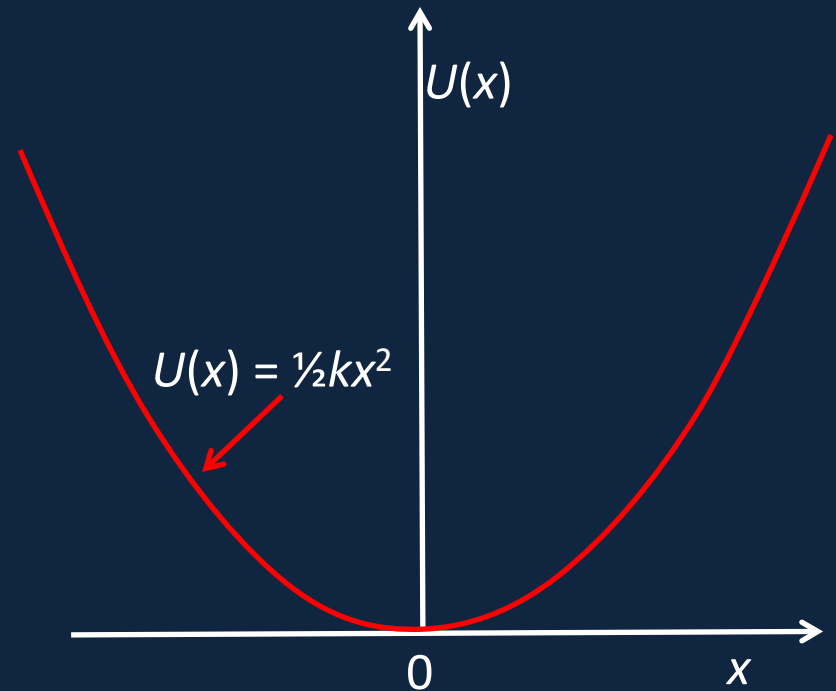
This work is stored in the spring as potential energy.

Potential Energy $U(x)$ Stored in Spring

- The potential energy curve is a **parabola**, its steepness determined by the spring constant k .
- For a mass m oscillating on the spring, with displacement

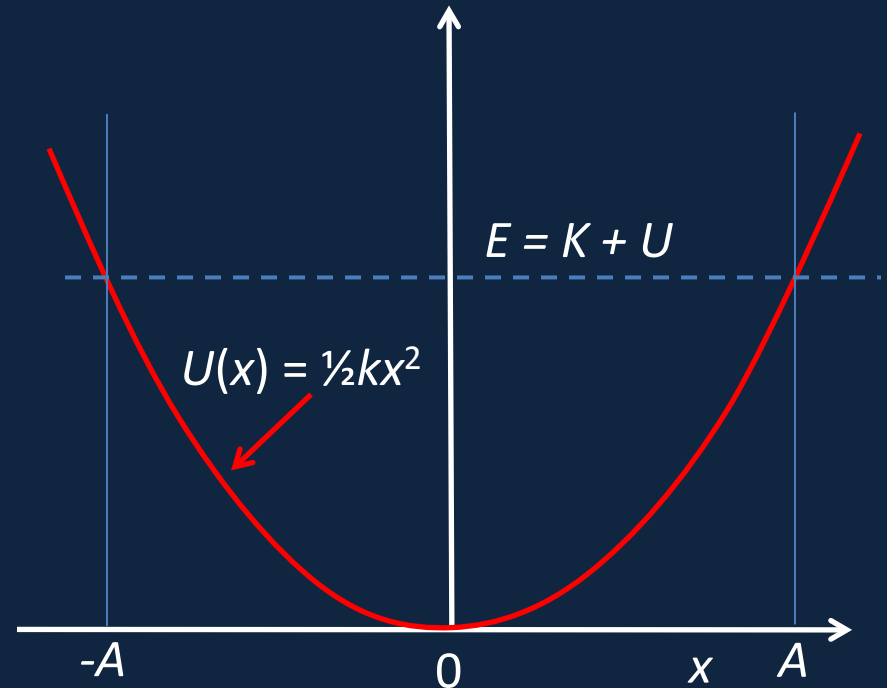
$$x = A \cos(\omega t + \phi)$$

the potential energy is $U(x) = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$



Total Energy E for a SHO

- The **total energy** E of a mass m oscillating on a spring having constant k is the **sum** of the mass's kinetic energy and the spring's potential energy:
 - $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$
 - For a given E , the mass will oscillate between the points $x = A$ and $-A$, where
 - $E = \frac{1}{2}kA^2$
 - Maximum speed is at $x = 0$, where $U(x) = 0$, and
 - $E = \frac{1}{2}mv^2$ at $x = 0$



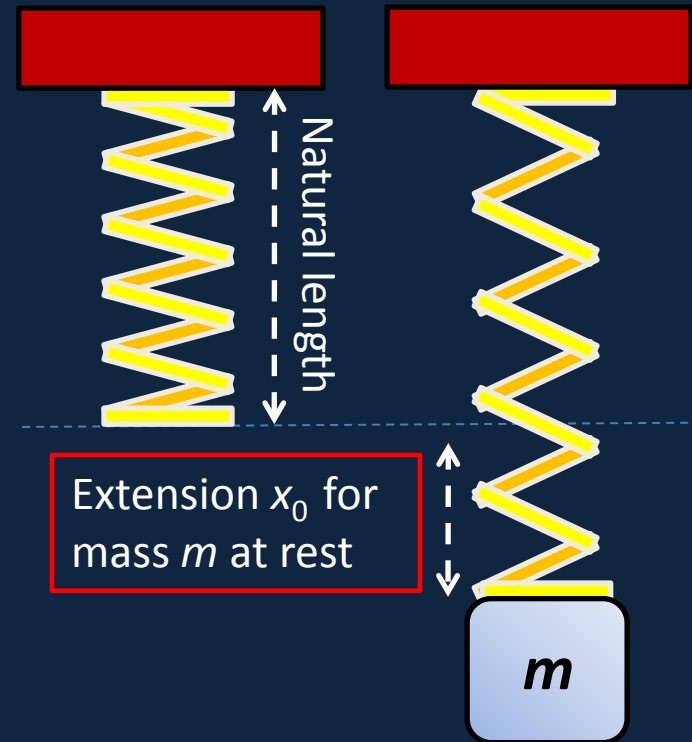
Mass *Hanging* on a Spring

- Suppose as before the spring constant is k .
- There will be an **extension x_0** , $kx_0 = mg$, when the **mass is at rest**.
- The equation of motion is now:

$$m \frac{d^2 x}{dt^2} = -k(x - x_0)$$

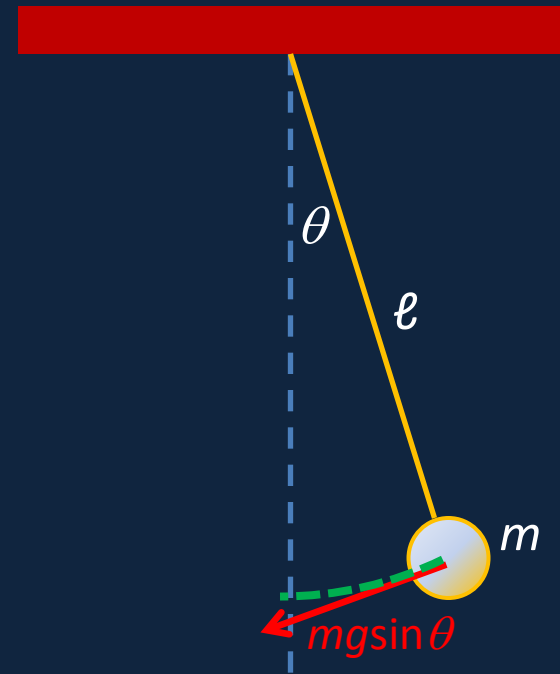
- with solution

$$x - x_0 = A \cos(\omega t + \phi), \quad \omega^2 = k / m.$$



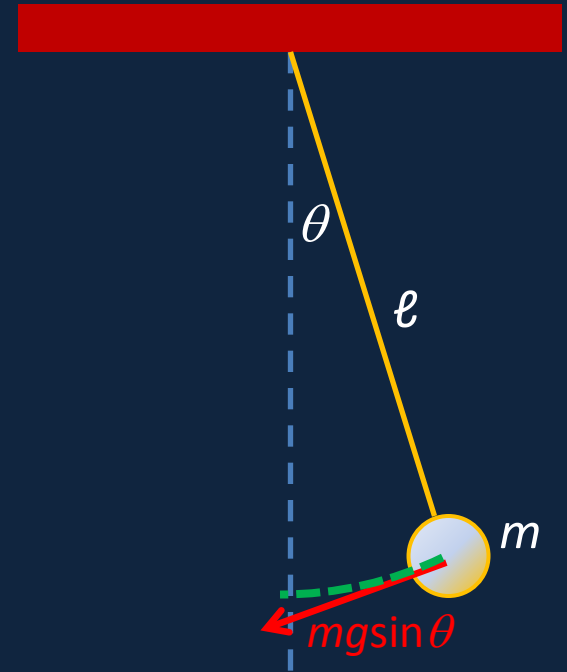
The Simple Pendulum

- A simple pendulum has a **bob**, a mass m treated as a **point mass**, at the end of a light string of length ℓ .
- We consider only small amplitude oscillations, and measure the displacement $x = \ell\theta$ along the **circular arc**.
- The restoring force is $F = -mg\sin\theta \cong -mg\theta$ along the arc.



$F = ma$ for the Simple Pendulum

- The displacement along the circular arc is $x = \ell\theta$.
- The restoring force is $F = -mg\sin\theta \cong -mg\theta = -mgx/\ell$ along the arc.
- $F = ma$ is
$$d^2x/dt^2 = -gx/\ell$$
 (canceling out m from both sides!).



Period of the Simple Pendulum

- The equation of motion

$$d^2x / dt^2 = -gx / \ell$$

has solution

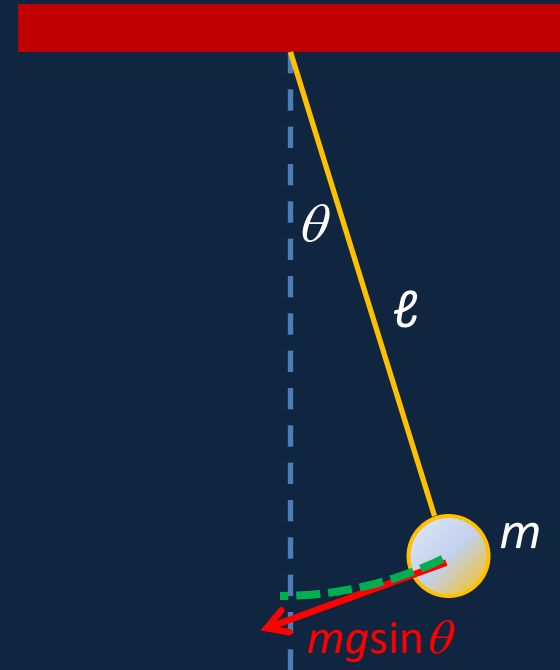
$$x = A \cos(\omega t + \phi)$$

- Here

$$\omega = \sqrt{g / \ell}$$

and the time for a complete swing

$$T = 2\pi / \omega = 2\pi \sqrt{\ell / g}.$$



The time for a complete swing doesn't depend on the mass m , for the same reason that different masses fall at the same rate.

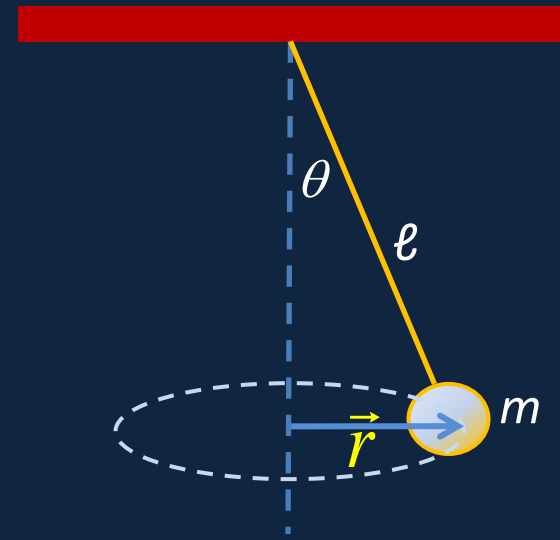
Reminder: the Conical Pendulum

- Imagine a conical pendulum in steady circular motion with small angle θ .
- As viewed from above, it moves in a circle, the centripetal force being $-(mg / \ell)\vec{r}$.
- So the equation of motion is

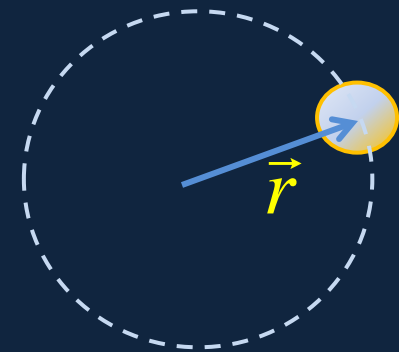
$$d^2\vec{r} / dt^2 = -(g / \ell)\vec{r}$$

and for the x -component of \vec{r}

$$d^2x / dt^2 = -gx / \ell$$



Top View:



The SHO and Circular Motion

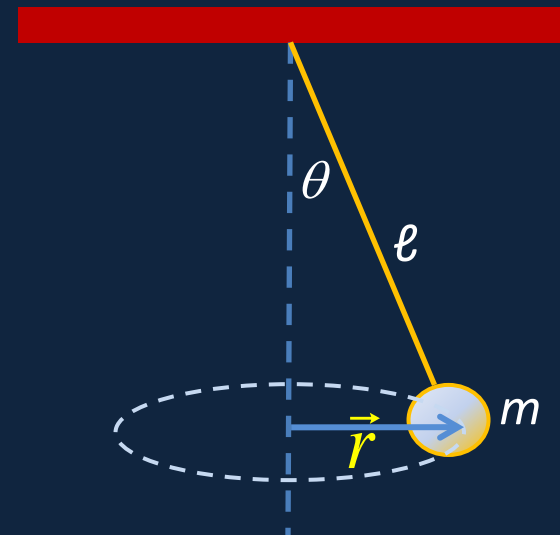
- We can now see that the equation of motion of the simple pendulum at small angles—which is a simple harmonic oscillator

$$d^2x / dt^2 = -gx / \ell$$

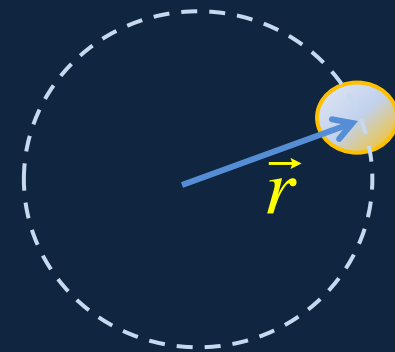
is nothing but the **x-component** of the steady **circular** motion of the conical pendulum

$$d^2\vec{r} / dt^2 = -(g / \ell)\vec{r}$$

- The simple pendulum is the **shadow** of the conical pendulum!



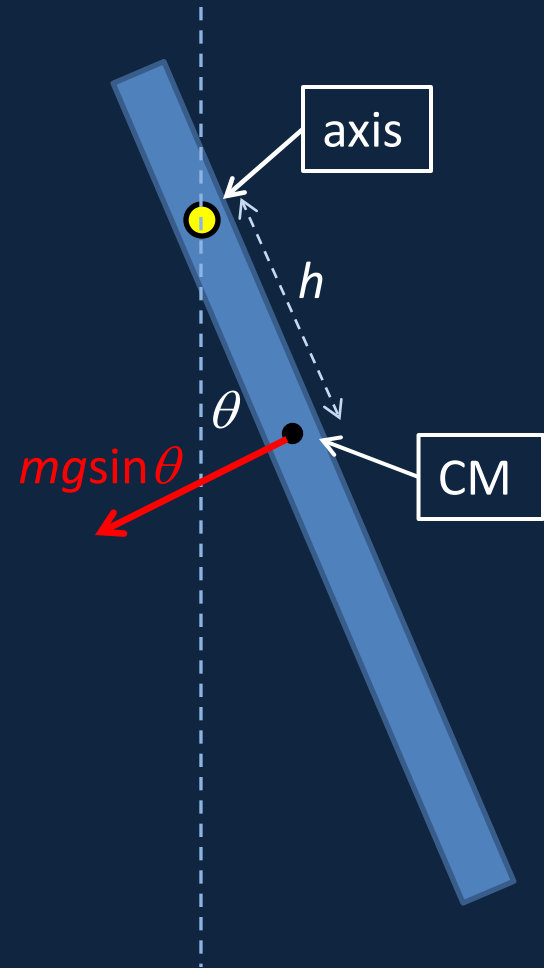
Top View:



The Physical Pendulum

- The term “physical pendulum” is used to denote a rigid body free to rotate about a fixed axis, making small angular oscillations under gravity.
- Taking the distance of the CM from the axis to be h , at (small) angle displacement θ , the torque is

$$\tau = mgh \sin \theta \cong mgh\theta$$



$\tau = I\alpha$ for the Physical Pendulum

- In the small angle approximation, the equation of motion $\tau = I\alpha$ is

$$I \frac{d^2\theta}{dt^2} = -mgh\theta$$

- with solution

$$\theta = \theta_0 \cos(\omega t + \phi)$$

- and

$$T = 2\pi / \omega = 2\pi \sqrt{I / mgh}.$$

- Remember this is $I_{\text{axis}} = I_{\text{CM}} + mh^2!$

