# More Angular Momentum

Physics 1425 Lecture 22

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# Torque as a Vector

- Suppose we have a wheel spinning about a fixed axis: then $\vec{\omega}$ always points along the axis—so $d\vec{\omega}/dt$ points along the axis too.
- If we want to write a vector equation

$$\vec{\tau} = I\vec{\alpha} = Id\vec{\omega}/dt$$

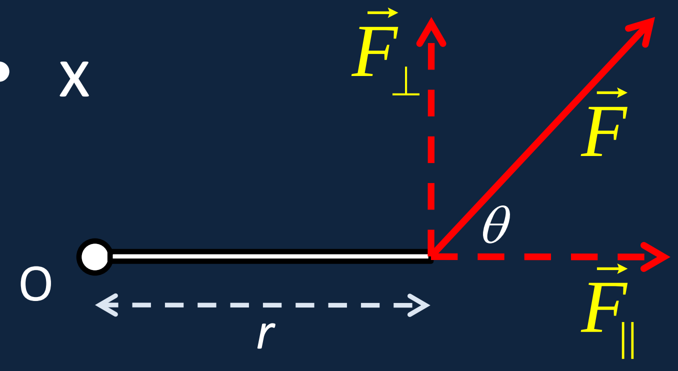
it's clear that the vector $\vec{\tau}$ is parallel to the vector $d\vec{\omega}/dt$: so $\vec{\tau}$ points along the axis too!

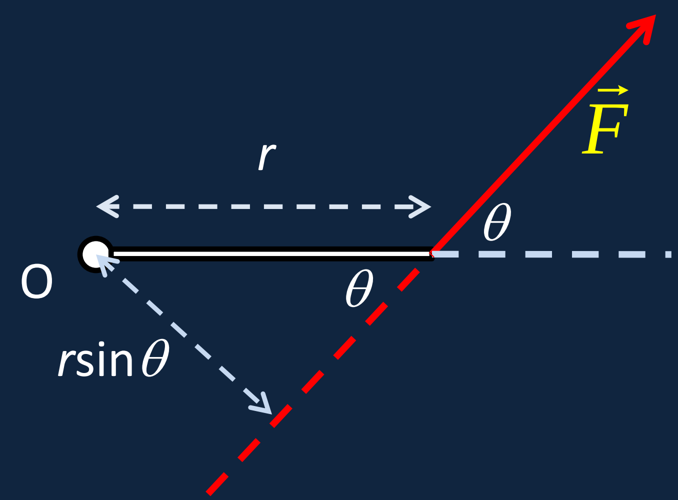
- BUT this vector $\vec{\tau}$, is, remember made of two other vectors: the force $\vec{F}$ and the place $\vec{r}$ where it acts!

# More Torque…

- Expressing the force vector $\vec{F}$ as a sum of components $\vec{F}_{\perp}$ (“fperp”) perpendicular to the lever arm and $\vec{F}_{\parallel}$ parallel to the arm, it’s clear that only $\vec{F}_{\perp}$ has leverage, that is, torque, about O.
  $\vec{F}_{\perp}$ has magnitude $F\sin\theta$, so $\tau = rF\sin\theta$.
- Alternatively, keep $\vec{F}$ and measure *its* lever arm about O: that’s $r\sin\theta$ .

- x

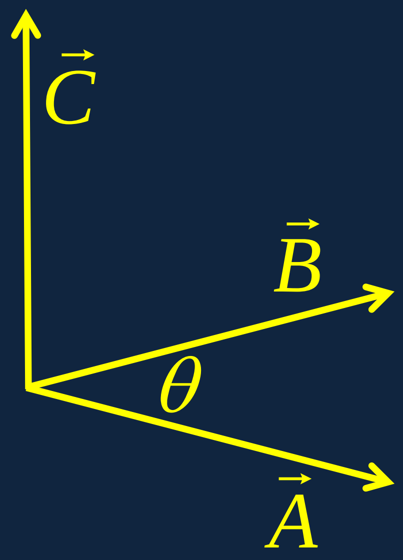


# Definition: The Vector Cross Product

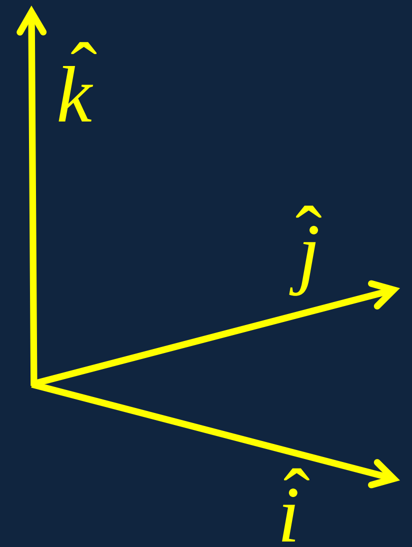
$$\vec{C} = \vec{A} \times \vec{B}$$

- The magnitude $C$ is $AB\sin\theta$, where $\theta$ is the angle between the vectors $\vec{A}, \vec{B}$.
- The direction of $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$, and is your right thumb direction if your curling fingers go from $\vec{A}$ to $\vec{B}$.



# The Vector Cross Product in Components

- Recall we defined the unit vectors $\hat{i}, \hat{j}, \hat{k}$ pointing along the *x*, *y*, *z* axes respectively, and a vector can be expressed as $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$



- Now $\hat{i} \times \hat{i} = 0, \; \hat{i} \times \hat{j} = \hat{k}, \; \hat{i} \times \hat{k} = -\hat{j}, \ldots$
- So

$$\vec{A} \times \vec{B} = \left( A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \right) \times \left( B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \right)$$

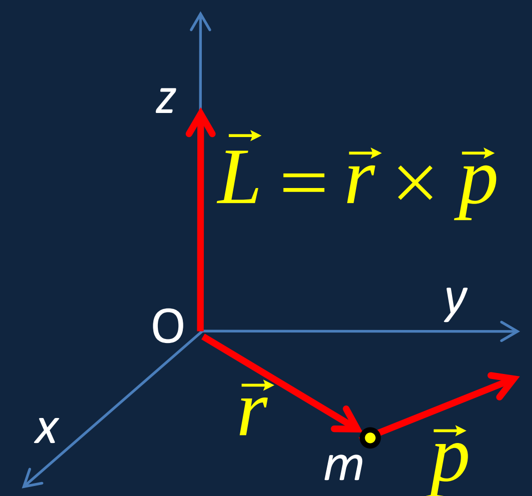
$$= \hat{i} \left( A_y B_z - A_z B_y \right) + \ldots$$

# Vector Angular Momentum of a Particle

- A particle with momentum $\vec{p}$ is at position $\vec{r}$ from the origin O.
- Its angular momentum about the origin is

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

- This is in line with our definition for part of a rigid body rotating about an axis: *but also works for a particle flying through space*.




Viewing the *x*-axis as coming out of the slide, this is a “right-handed” set of axes: $\hat{i} \times \hat{j} = +\hat{k}$

# Angular Momentum and Torque for a Particle

- Angular momentum about the origin of particle mass *m*, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

Torque about the origin

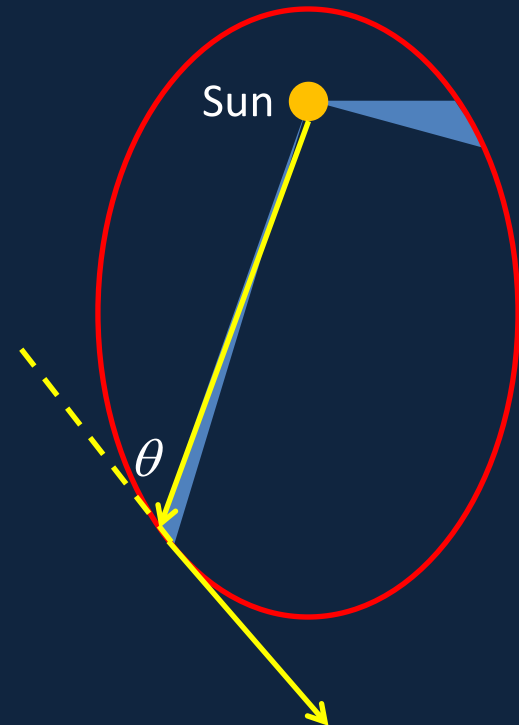
- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

# Kepler's Second Law

As the planet moves, a line from the planet to the center of the Sun **sweeps out equal areas in equal times.**

- In unit time, it moves through a distance $\vec{v}$.
- The area of the triangle swept out is $\frac{1}{2} r v \sin\theta$ (from ½ base x height)
- This is ½$L/m$, $\vec{L} = \vec{r} \times \vec{p}$.
- Kepler's Law is telling us <u>the angular momentum about the Sun is constant</u>: this is because the Sun's pull has *zero torque* about the Sun itself.




The **base** of the thin blue triangle is a distance *v* along the tangent. The **height** is the perp distance of this tangent from the Sun.

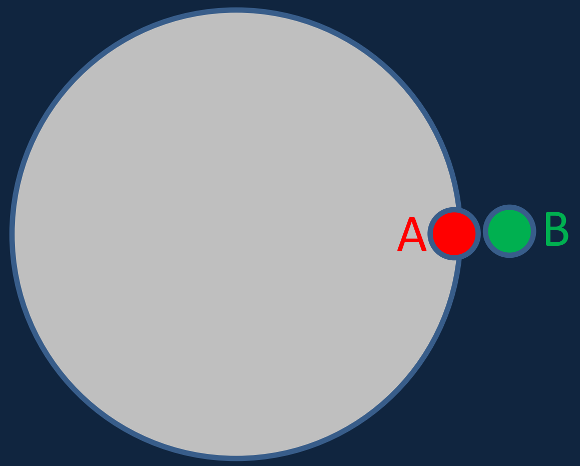
# Guy on Turntable

- A, of mass *m*, is standing on the edge of a frictionless turntable, a disk of mass 4*m*, radius *R*, next to B, who's on the ground.
- A now walks around the edge until he's back with B.
- How far does he walk?

A. $2\pi R$

B. $2.5\pi R$

C. $3\pi R$



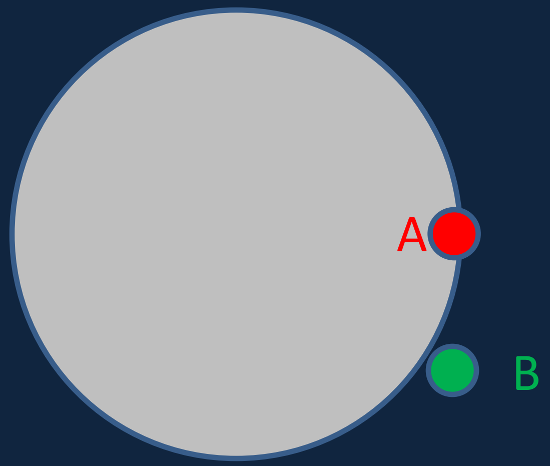
# Guy on Turntable Catches a Ball

- A, of mass *m*, is standing on the edge of a frictionless turntable, a disk of mass 4*m*, radius *R*, at rest.
- B, who's on the ground, throws a ball weighing 0.1*m* at speed *v* to A, who catches it without slipping.
- What is the angular momentum of turntable + man + ball now?

A. 0.1mvR

B. (0.1/3.1)mvR

C. (0.1/5.1)mvR



# Guy on Turntable Walks In

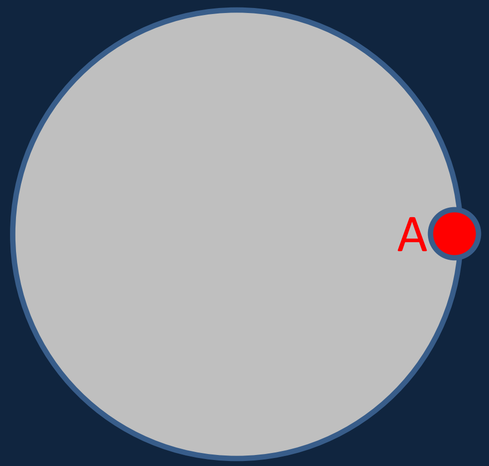
- A, of mass *m*, is standing on the edge of a frictionless turntable, a disk of mass 4*m*, radius *R*, which is rotating at 6 rpm.
- A walks to the exact center of the turntable.
- How fast (approximately) is the turntable now rotating?

A. 12 rpm

B. 9 rpm

C. 6 rpm

D. 4 rpm



# Reminder: Angular Momentum and Torque for a Particle…

- Angular momentum about the origin of particle mass *m*, momentum $\vec{p}$ at $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

# Lots of Particles

- Suppose we have particles acted on by external forces, and also acting on each other.
- The rate of change of angular momentum of one of the particles about a fixed origin O is:

$$d\vec{L}_i / dt = \vec{\tau}_{i\ \mathrm{int}} + \vec{\tau}_{i\ \mathrm{ext}}$$

- The internal torques come in equal and opposite pairs, so

$$d\vec{L} / dt = \sum_i d\vec{L}_i / dt = \sum_i \vec{\tau}_{i\ \mathrm{ext}}$$

# Rotational Motion of a Rigid Body

- For a collection of interacting particles, we've seen that

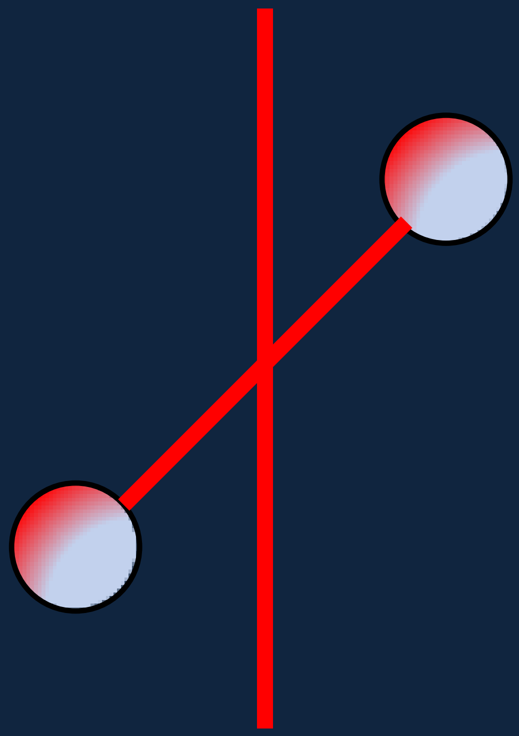
$$d\vec{L} / dt = \sum_i \vec{\tau}_i$$

  the vector sum of the applied torques, $\vec{L}$ and the $\vec{\tau}_i$ being measured about a fixed origin O.
- A rigid body is equivalent to a set of connected particles, so the same equation holds.
- It is also true (proof in book) that even if the CM is accelerating,

$$\boxed{d\vec{L}_{\mathrm{CM}} / dt = \sum \vec{\tau}_{\mathrm{CM}}}$$

# Angular Velocity and Angular Momentum Need not be Parallel

- Imagine a dumbbell attached at its center of mass to a light vertical rod as shown, then the system rotates about the vertical line.
- The angular velocity vector $\vec{\omega}$ is vertical.
- The total angular momentum $\vec{L}$ about the CM is $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$.
- Think about this at the instant the balls are in the plane of the slide—so is $\vec{L}$, but it's not vertical!



# When *are* Angular Velocity and Angular Momentum Parallel?

- When the rotating object is symmetric about the axis of rotation: if for each mass on one side of the axis, there's an equal mass at the corresponding point on the other side.
- For this pair of masses, $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$ *is* along the axis.
- (Check it out!)