

More Angular Momentum

Physics 1425 Lecture 22

Torque as a Vector

- Suppose we have a wheel spinning about a fixed axis: then $\vec{\omega}$ always points along the axis—so $d\vec{\omega}/dt$ points along the axis too.

- If we want to write a vector equation

$$\vec{\tau} = I\vec{\alpha} = Id\vec{\omega}/dt$$

it's clear that the vector $\vec{\tau}$ is parallel to the vector $d\vec{\omega}/dt$: so $\vec{\tau}$ points along the axis too!

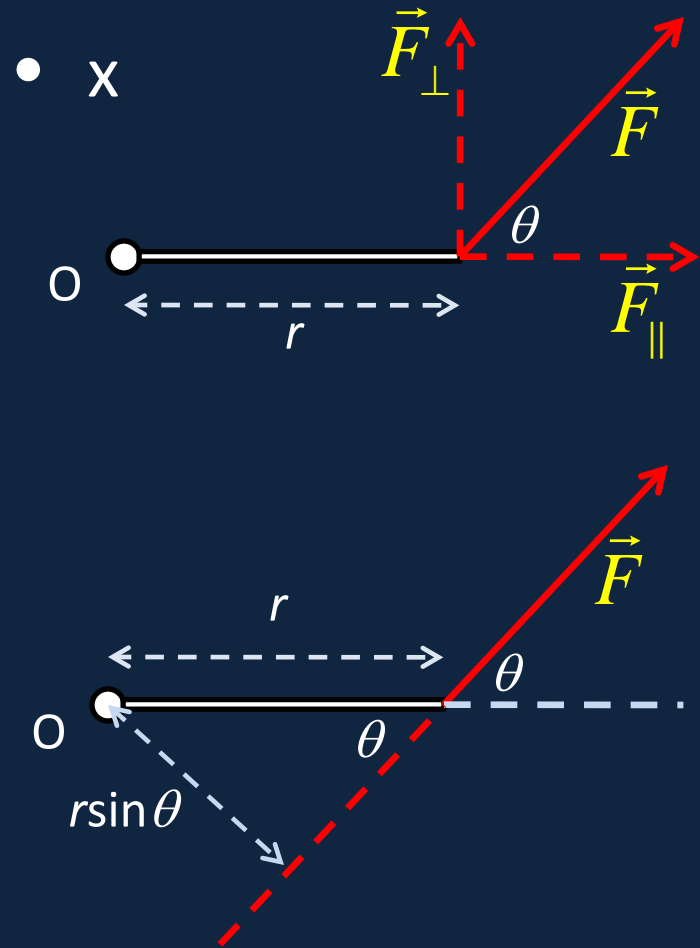
- **BUT** this vector $\vec{\tau}$, is, remember made of two other vectors: the force \vec{F} and the place \vec{r} where it acts!

More Torque...

- Expressing the force vector \vec{F} as a sum of components \vec{F}_\perp (“fperp”) perpendicular to the lever arm and \vec{F}_\parallel parallel to the arm, it’s clear that only \vec{F}_\perp has leverage, that is, torque, about O.

\vec{F}_\perp has magnitude $F\sin\theta$, so $\tau = rF\sin\theta$.

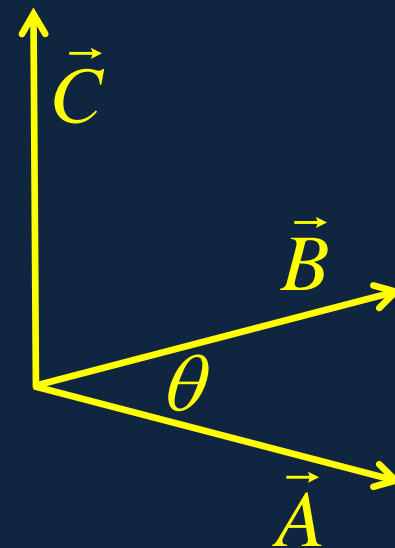
- Alternatively, keep \vec{F} and measure *its* lever arm about O: that’s $r\sin\theta$.



Definition: The Vector Cross Product

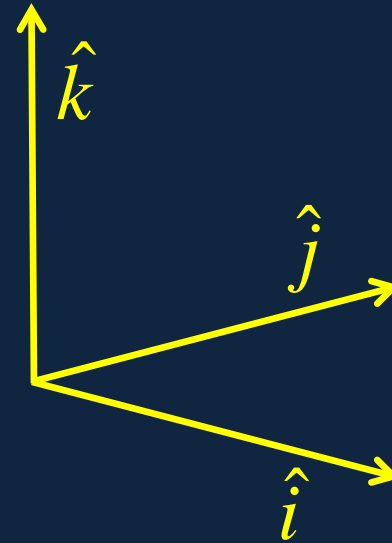
$$\vec{C} = \vec{A} \times \vec{B}$$

- The **magnitude** C is $AB\sin\theta$, where θ is the angle between the vectors \vec{A}, \vec{B} .
- The **direction** of \vec{C} is perpendicular to both \vec{A} and \vec{B} , and is your right thumb direction if your curling fingers go from \vec{A} to \vec{B} .



The Vector Cross Product in Components

- Recall we defined the unit vectors $\hat{i}, \hat{j}, \hat{k}$ pointing along the x, y, z axes respectively, and a vector can be expressed as $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$



- Now $\hat{i} \times \hat{i} = 0, \hat{i} \times \hat{j} = \hat{k}, \hat{i} \times \hat{k} = -\hat{j}, \dots$
- So

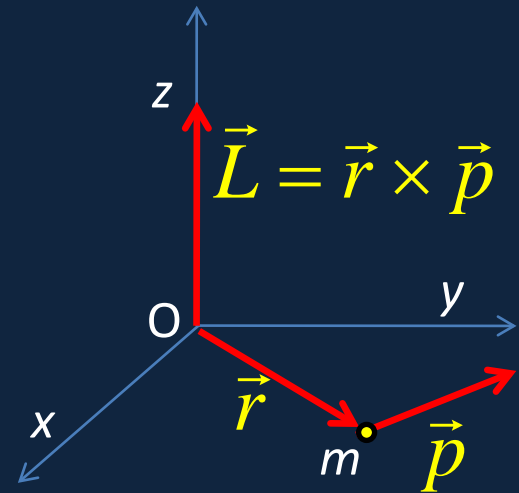
$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\ &= \hat{i} (A_y B_z - A_z B_y) + \dots\end{aligned}$$

Vector Angular Momentum of a Particle

- A particle with momentum \vec{p} is at position \vec{r} from the origin O.
- Its angular momentum about the origin is

$$\vec{L} = \vec{r} \times \vec{p}$$

- This is in line with our definition for part of a rigid body rotating about an axis: *but also works for a particle flying through space.*



Viewing the x-axis as coming out of the slide, this is a “right-handed” set of axes:

$$\hat{i} \times \hat{j} = +\hat{k}$$

Angular Momentum and Torque for a Particle

- Angular momentum about the origin of particle mass m , momentum \vec{p} at \vec{r}

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

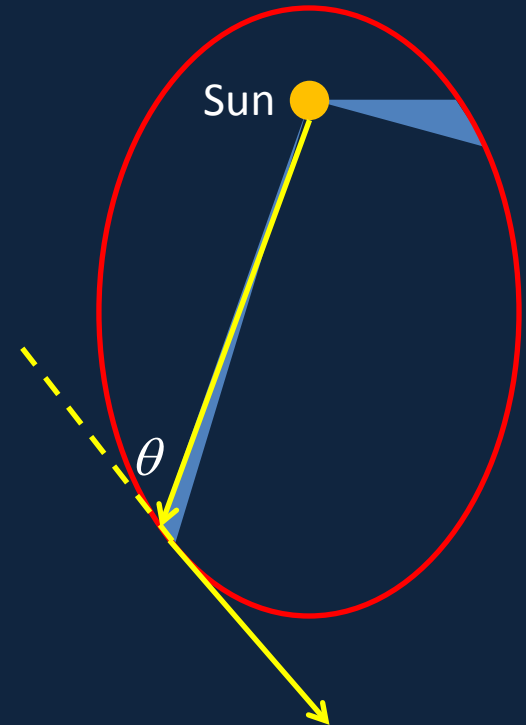
Torque about the origin



Kepler's Second Law

As the planet moves, a line from the planet to the center of the Sun **sweeps out equal areas in equal times.**

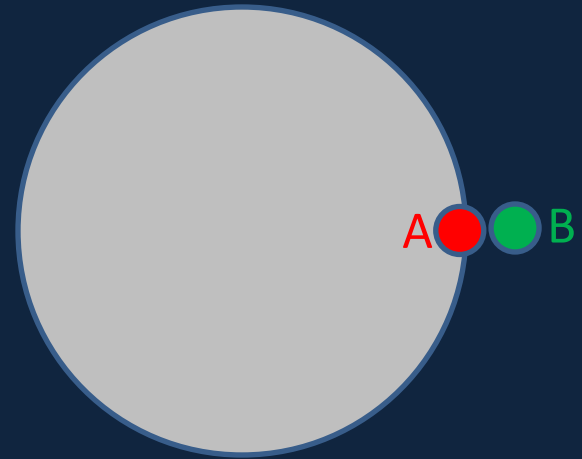
- In unit time, it moves through a distance \vec{v} .
- The area of the triangle swept out is $\frac{1}{2}rv\sin\theta$ (from $\frac{1}{2}$ base x height)
- This is $\frac{1}{2}L/m$, $\vec{L} = \vec{r} \times \vec{p}$.
- Kepler's Law is telling us the angular momentum about the Sun is constant: this is because the Sun's pull has *zero torque* about the Sun itself.



The **base** of the thin blue triangle is a distance v along the tangent. The **height** is the perp distance of this tangent from the Sun.

Guy on Turntable

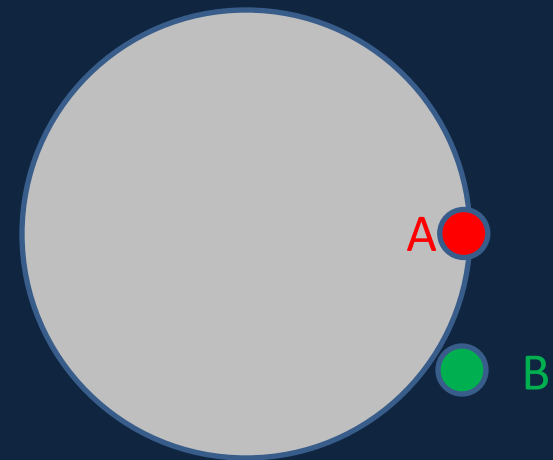
- **A**, of mass m , is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R , next to **B**, who's on the ground.
- **A** now walks around the edge until he's back with **B**.
- How far does he walk?
 - A. $2\pi R$
 - B. $2.5\pi R$
 - C. $3\pi R$



Guy on Turntable Catches a Ball

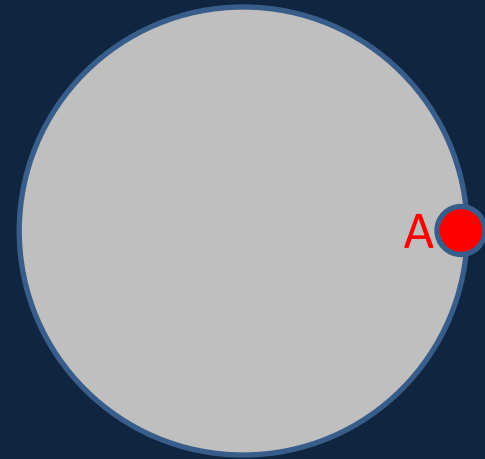
- **A**, of mass m , is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R , **at rest**.
- **B**, who's on the ground, throws a ball weighing $0.1m$ at speed v to **A**, who catches it without slipping.
- What is the angular momentum of turntable + **man** + ball now?

- A. $0.1mvR$
- B. $(0.1/3.1)mvR$
- C. $(0.1/5.1)mvR$



Guy on Turntable Walks In

- A , of mass m , is standing on the edge of a frictionless turntable, a disk of mass $4m$, radius R , which is rotating at 6 rpm.
- A walks to the exact center of the turntable.
- How fast (approximately) is the turntable now rotating?
 - A. 12 rpm
 - B. 9 rpm
 - C. 6 rpm
 - D. 4 rpm



Reminder: Angular Momentum and Torque for a Particle...

- Angular momentum about the origin of particle mass m , momentum \vec{p} at \vec{r}

$$\vec{L} = \vec{r} \times \vec{p}$$

- Rate of change:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

Lots of Particles

- Suppose we have particles acted on by external forces, and also acting on each other.
- The rate of change of angular momentum of one of the particles about a fixed origin O is:

$$d\vec{L}_i / dt = \vec{\tau}_{i \text{ int}} + \vec{\tau}_{i \text{ ext}}$$

- The internal torques come in equal and opposite pairs, so

$$d\vec{L} / dt = \sum_i d\vec{L}_i / dt = \sum_i \vec{\tau}_{i \text{ ext}}$$

Rotational Motion of a Rigid Body

- For a collection of interacting particles, we've seen that

$$d\vec{L} / dt = \sum_i \vec{\tau}_i$$

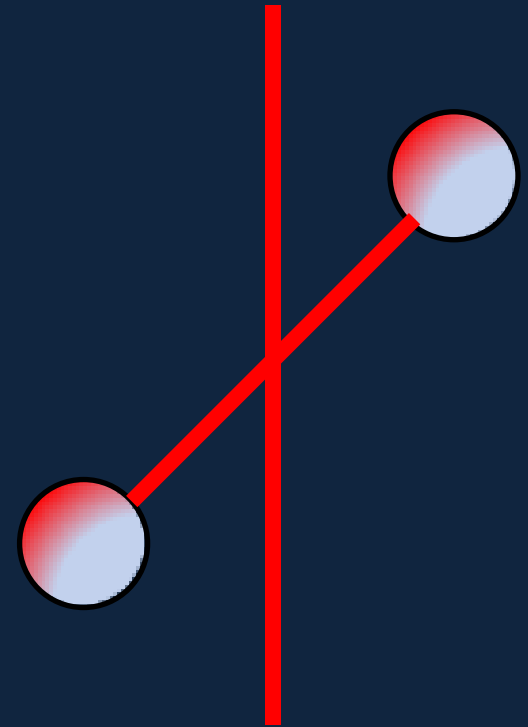
the vector sum of the applied torques, \vec{L} and the $\vec{\tau}_i$, being measured about a fixed origin O.

- A rigid body is equivalent to a set of connected particles, so the same equation holds.
- It is also true (proof in book) that even if the CM is accelerating,

$$d\vec{L}_{\text{CM}} / dt = \sum \vec{\tau}_{\text{CM}}$$

Angular Velocity and Angular Momentum Need not be Parallel

- Imagine a dumbbell attached at its center of mass to a light vertical rod as shown, then the system rotates about the vertical line.
- The angular velocity vector $\vec{\omega}$ is vertical.
- The total angular momentum \vec{L} about the CM is $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$.
- Think about this at the instant the balls are in the plane of the slide—so is \vec{L} , but it's not vertical!



When *are* Angular Velocity and Angular Momentum Parallel?

- When the rotating object is symmetric about the axis of rotation: if for each mass on one side of the axis, there's an equal mass at the corresponding point on the other side.
- For this pair of masses, $\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$ is along the axis.
- (Check it out!)

