

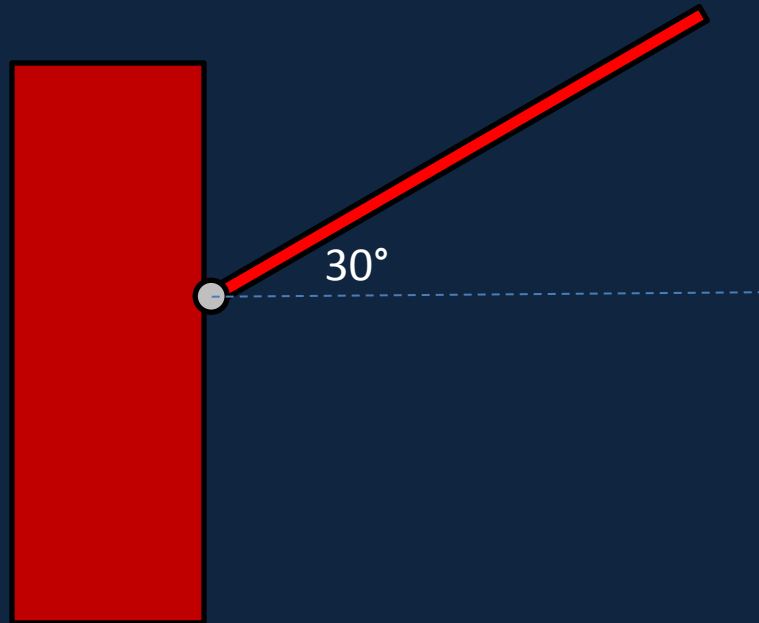
More Rotational Dynamics

Physics 1425 Lecture 20

Clicker Question

A uniform rod is free to rotate in a vertical plane about a frictionless hinge at one end. It is released from rest at an angle of 30° . ($I = (1/3)ML^2$, $\tau = Mg(L/2)\cos 30^\circ$)
The initial **downward acceleration of the free end** of the rod is:

- A. equal to g
- B. greater than g
- C. less than g



Rotational Kinetic Energy

- Imagine a rotating body as composed of many small masses m_i at distances r_i from the axis of rotation.
- The mass m_i has speed $v = \omega r_i$, so $KE = \frac{1}{2}m_i r_i^2 \omega^2$.
- The total KE of the rotating body (**assuming the axis is at rest**) is

$$K = \sum_i \left(\frac{1}{2} m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2$$

Torque Power

- If a net torque τ is acting on a rotating body, the net power is the rate of change of rotational energy

$$\frac{d}{dt} \left(\frac{1}{2} I \omega^2 \right) = I \omega \frac{d\omega}{dt} = I \omega \alpha = \omega \tau \quad (\text{recall } \tau = I \alpha)$$

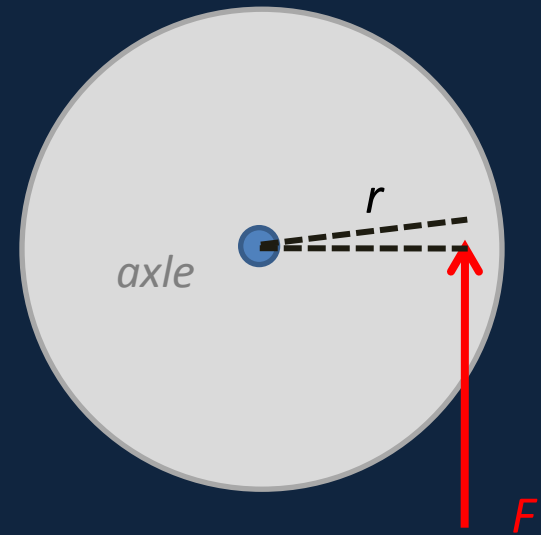
- So the **rate of working** of the torque, **power** = $\tau \omega$, its value x the angular velocity.
- Total work done over some time period is

$$\int \tau \omega dt = \int \tau \frac{d\theta}{dt} dt = \int \tau d\theta$$

- This is just like $\int F dx$ in linear motion.

Work Done by a Torque

- Suppose the torque is a force F acting at a distance r from the center as shown. If the disk turns through an angle $d\theta$, the force acts through a distance $ds = rd\theta$ so does work $Fds = Frd\theta$.

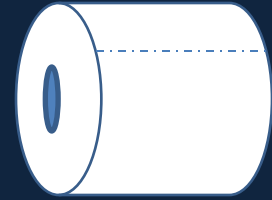


- But $\tau = rF$, so the work

$$Fds = Frd\theta = \tau d\theta$$

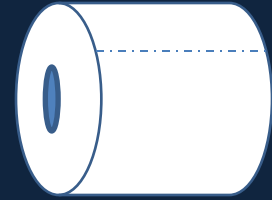
Force x distance = torque x angle

A Familiar Item...

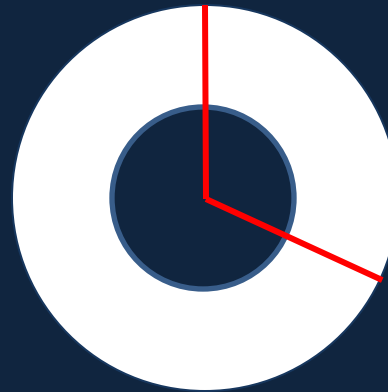


- A roll of toilet paper has diameter 0.1m, which happens also to be the length of one sheet.
- What is the angle *in radians* subtended at the central line of the roll by one sheet in the outside layer?
- A. 1
- B. 2
- C. 0.5
- D. π
- E. $1/\pi$

A Familiar Item...

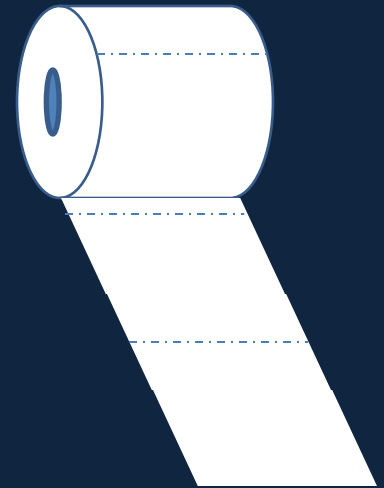


- A roll of toilet paper has diameter 0.1m, which happens also to be the length of one sheet.
- What is the angle *in radians* subtended at the central line of the roll by one sheet in the outside layer?
- It's about 2 radians:



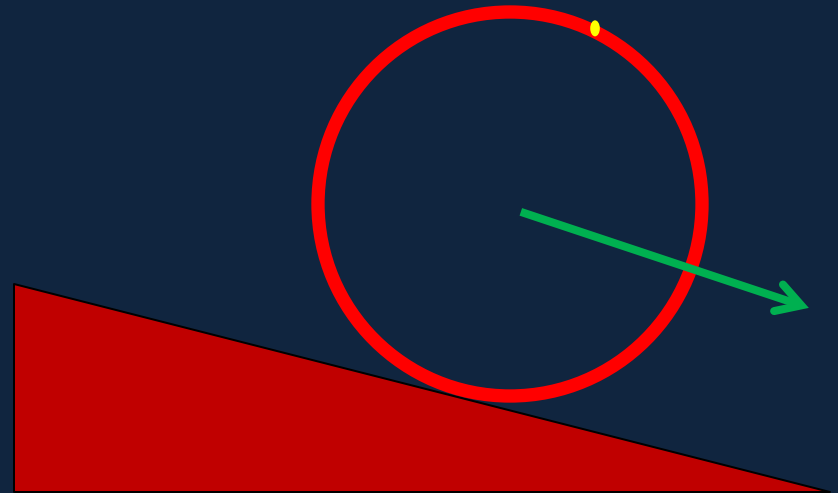
On a Roll...

- This roll (0.1 m diameter, 0.1 m sheets) rolls across the table, unwinding three sheets per second.
- Give its CM velocity, *and* the angular velocity about the CM in radians/sec.
 - A. 0.3, 6
 - B. 0.3, 3
 - C. 0.6, 6
 - D. 0.3, 3π



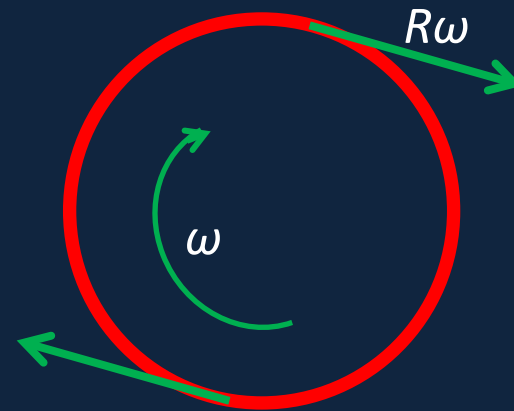
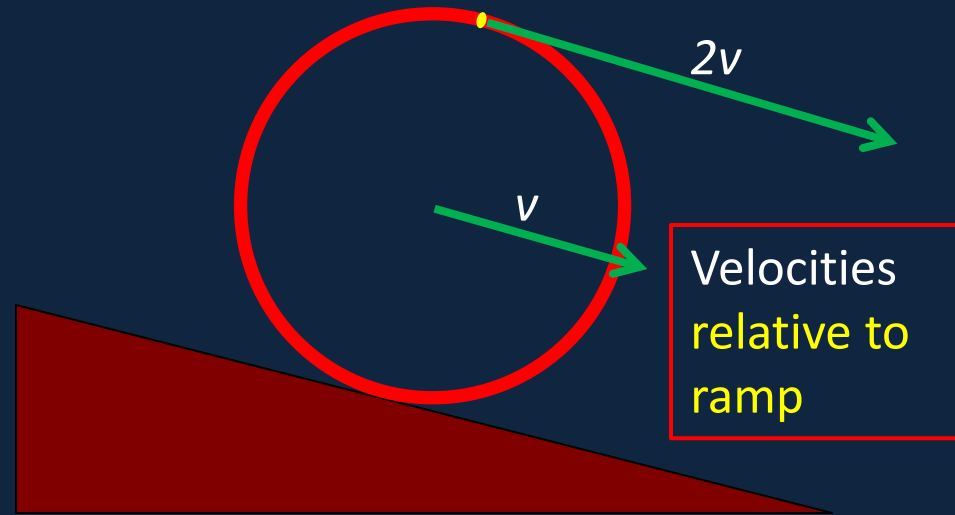
Clicker Question

- A hoop is rolling down a ramp (without slipping) at v m/sec.
- How fast is the **point** on the hoop **furthest** from the ramp moving?
- A. v m/sec
- B. $2v$ m/sec
- C. $4v$ m/sec



Hoop Rolling Down Ramp

- If there's no slipping, the point on the hoop in contact with the ramp is at rest—the hoop is at that instant rotating about that point.
- So if the center is moving at v , the “top” point is moving at $2v$.
- Relative to the center, all points are moving at speed $R\omega$ tangentially.
- Hence, since the bottom's at rest: $v = R\omega$
- The “no slip” condition.



Velocities relative to center of hoop

Total Kinetic Energy of Rolling Hoop

- Suppose as usual the hoop is made of many small masses m_i and the mass m_i is moving at \vec{v}_i . Then the **total KE** is $\sum_i \frac{1}{2} m_i \vec{v}_i^2$.
- This total kinetic energy depends on **both** the **translational motion** (the center of the hoop is moving) **and** the hoop's **rotation** about the center.
- How do we sort this out?

Separating Translational and Rotational Kinetic Energies: Details

- Suppose we have **rigid body** we represent as a collection of masses m_i , with individual velocities \vec{v}_i .
- Let's suppose the CM is moving at \vec{v}_{CM} , so the total **linear** momentum is $M \vec{v}_{\text{CM}}$, M being the total mass.
- To **separate out the rotational motion**, we'll write the individual velocities $\vec{v}_i = \vec{v}_{\text{CM}} + \vec{u}_i$: so \vec{u}_i is velocity of m_i relative to the CM.
- Then the total kinetic energy is

$$\sum_i \frac{1}{2} m_i \vec{v}_i^2 = \sum_i \frac{1}{2} m_i (\vec{v}_{\text{CM}} + \vec{u}_i)^2 = \frac{1}{2} M \vec{v}_{\text{CM}}^2 + \vec{v}_{\text{CM}} \cdot \sum_i m_i \vec{u}_i + \sum_i \frac{1}{2} m_i \vec{u}_i^2$$

$$KE = \frac{1}{2} M \vec{v}_{\text{CM}}^2 + \frac{1}{2} I_{\text{CM}} \omega^2$$

- Because relative to the CM $\sum_i m_i \vec{u}_i = \frac{d}{dt} \sum_i m_i \vec{r}_i = 0$, $\vec{u}_i^2 = r_i^2 \omega^2$.

Total Energy: the Bottom Line

- In case the last slide was too much, what you *really* need is that the **total kinetic energy** of a moving, rotating object **is a sum of two terms**:
- **Translational KE**, the same as if all the mass is moving with the velocity of the center of mass, **and**
- **Rotational KE**, about the center of mass:

$$KE = \frac{1}{2} M \vec{v}_{\text{CM}}^2 + \frac{1}{2} I_{\text{CM}} \omega^2$$

How Fast Does a Hoop Roll Down a Ramp?

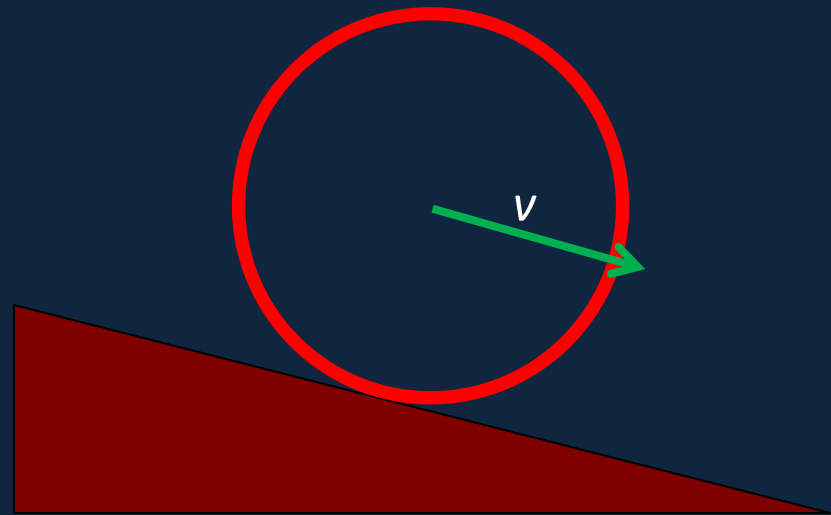
- Assuming no slipping, so

$$v = R\omega$$

- The total kinetic energy at an instant:

$$\begin{aligned} KE &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}(mR^2)\omega^2 \\ &= mv^2. \end{aligned}$$

- If it's rolled down through height h from a standing start, $mv^2 = mgh$, so $v = \sqrt{gh}$
- For a frictionless sliding mass, $\frac{1}{2}mv^2 = mgh$, so $v = \sqrt{2gh}$: faster!



The hoop takes **longer** to get down than a low-friction sliding block, because the **same** loss in potential energy has to supply **BOTH** translational *KE* and rotational *KE* for the hoop.

Ramp Race

A hoop, a solid cylinder and a solid sphere roll down the same ramp from a standing start. Who clocks the fastest time?

- A. The hoop
- B. The solid cylinder
- C. The solid sphere
- D. It depends on the sizes and/or masses.

A New Look for $\tau = I\alpha$

- We've seen how $\tau = I\alpha$ works for a body rotating about a **fixed axis**.
- $\tau = I\alpha$ is not true in general if the axis of rotation is *itself* accelerating
- **BUT it IS true if the axis is through the CM, and isn't changing direction!**
- This is quite tricky to prove—it's in the book
- And $\tau_{\text{CM}} = I_{\text{CM}}\alpha_{\text{CM}}$ is often useful, as we'll see.