Rotational Dynamics

- **Newton’s First Law**: a rotating body will continue to rotate at constant angular velocity as long as there is no torque acting on it.
- Picture a grindstone on a smooth axle.
- BUT the axle must be *exactly* at the center of gravity—otherwise gravity will provide a torque, and the rotation will not be at constant velocity!
How is Angular Acceleration Related to Torque?

• Think about a tangential force $F$ applied to a mass $m$ attached to a light disk which can rotate about a fixed axis. (A radially directed force has zero torque, does nothing.)

• The relevant equations are:

$$F = ma, \quad \alpha = r\alpha, \quad \tau = rF.$$ 

• Therefore $F = ma$ becomes

$$\tau = mr^2 \alpha$$
Newton’s Second Law for Rotations

• For the special case of a mass $m$ constrained by a light disk to circle around an axle, the angular acceleration $\alpha$ is proportional to the torque $\tau$ exactly as in the linear case the acceleration $a$ is proportional to the force $F$:
  
  \[ \tau = mr^2\alpha \quad \text{and} \quad F = ma \]

The angular equivalent of inertial mass $m$ is the moment of inertia $mr^2$. 
• Suppose now a light disk has several different masses attached at different places, and various forces act on them. As before, radial components cause no rotation, we have a sum of torques.
• BUT the rigidity of the disk ensures that a force applied to one mass will cause a torque on the others!
• How do we handle that?

\[ F_1 \times r_1 = \tau_{\text{net}} \]

\[ F_2 \times r_2 = \tau_{\text{net}} \]
Newton’s Third Law for a Rigid Rotating Body

• If a rigid body is made up of many masses $m_i$ connected by rigid rods, the force exerted along the rod of $m_i$ on $m_j$ is equal in magnitude, opposite in direction and along the same line as that of $m_j$ on $m_i$, therefore the internal torques come in equal and opposite pairs, and cannot contribute to the body’s angular acceleration.

• It follows that the angular acceleration is generated by the sum of the external torques.
Moment of Inertia of a Solid Body

- Consider a flat square plate rotating about a perpendicular axis with angular acceleration $\alpha$. One small part of it, $\Delta m_i$, distance $r_i$ from the axle, has equation of motion

$$\tau_i = \tau_i^{\text{ext}} + \tau_i^{\text{int}} = \Delta m_i r_i^2 \alpha$$

- Adding contributions from all parts of the wheel

$$\tau = \sum_i \tau_i^{\text{ext}} = \left( \sum_i \Delta m_i r_i^2 \right) \alpha = I \alpha$$

- $I$ is the Moment of Inertia.
Calculating Moments of Inertia

- A thin hoop of radius $R$ (think a bicycle wheel) has all the mass distance $R$ from a perpendicular axle through its center, so its moment of inertia is
  \[ I = \sum_i \Delta m_i r_i^2 = MR^2 \]

- A uniform rod of mass $M$, length $L$, has moment of inertia about one end
  \[ I = \int_0^L x^2 \frac{M}{L} dx = \frac{1}{3} ML^2 \]

Mass of length $dx$ of rod is $(M/L)dx$
Disks and Cylinders

- **A disk**: mass $M$, radius $R$, is a sum of nested rings.
- The *red ring*, radius $r$ and thickness $dr$, has area $2\pi r dr$, hence mass $dm = M(2\pi r dr / \pi R^2)$.
- Adding up rings to make a disk,
  \[
  I = \int_0^R r^2 dm = \int_0^R r^2 \left( \frac{2M}{R^2} \right) r dr = \frac{1}{2} MR^2
  \]
- **A cylinder** is just a stack of disks, so it’s also $\frac{1}{2} MR^2$ about the axle.
Parallel Axis Theorem

- If we already know $I_{CM}$ about some line through the CM (we take it as the z-axis), then $I$ about a parallel line at a distance $h$ is

\[ I = I_{CM} + Mh^2 \]

- Here’s the proof:

\[
I = \sum_i m_i \vec{r}_i^2 = \sum_i m_i \left( \vec{r}_i' + \vec{h} \right)^2
\]

\[
= \sum_i m_i \vec{r}_i'^2 + 2 \vec{h} \cdot \sum_i m_i \vec{r}_i' + M\vec{h}^2
\]

\[
= I_{CM} + Mh^2 \quad \text{(Since } \sum_i m_i \vec{r}_i' = 0.\text{)}
\]

Moment of inertia $I$ about perpendicular axis through $A$

- We prove it for a 2D object—the proof in 3D is exactly the same, taking the line through the CM as the z-axis.
Clicker Question

We found the moment of inertia of a rod about a perpendicular line through one end was \( \frac{1}{3} ML^2 \). Use the parallel axis theorem to figure out what it is about a perpendicular line through the center of the rod.

A \( \frac{1}{3} ML^2 \)
B \( \frac{7}{12} ML^2 \)
C \( \frac{1}{2} ML^2 \)
D \( \frac{1}{4} ML^2 \)
E \( \frac{1}{12} ML^2 \)
Perpendicular Axis Theorem

• For a 2D object (a thin plate) the moment of inertia $I_z$ about a perpendicular axis equals the sum of the moments of inertia about any two axes at right angles through the same point in the plane,

$$I_z = I_x + I_y$$

• Proof:

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i \left(x_i^2 + y_i^2\right) = I_x + I_y$$
Clicker Question

Given that the moment of inertia of a disk about its axle is $\frac{1}{2} MR^2$, use the perpendicular axis theorem to find the moment of inertia of a disk about a line through its center and in its plane.

A $\frac{1}{2} MR^2$

B $\frac{1}{4} MR^2$

C $MR^2$
Rotational Kinetic Energy

- Imagine a rotating body as composed of many small masses $m_i$ at distances $r_i$ from the axis of rotation.
- The mass $m_i$ has speed $v = \omega r_i$, so $KE = \frac{1}{2} m_i r_i^2 \omega^2$.
- The total $KE$ of the rotating body (assuming the axis is at rest) is

$$K = \sum_i \left( \frac{1}{2} m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2$$