Kinetic Energy and Energy Conservation

Physics 1425 Lecture 13

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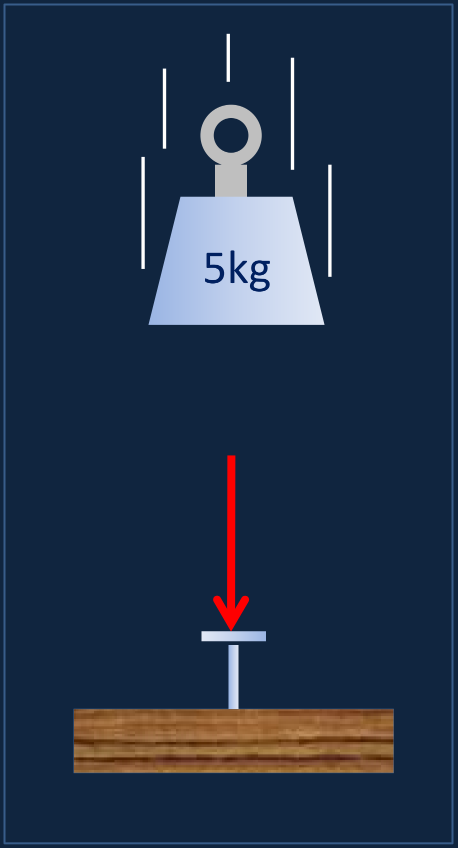
Moving Things Have Energy

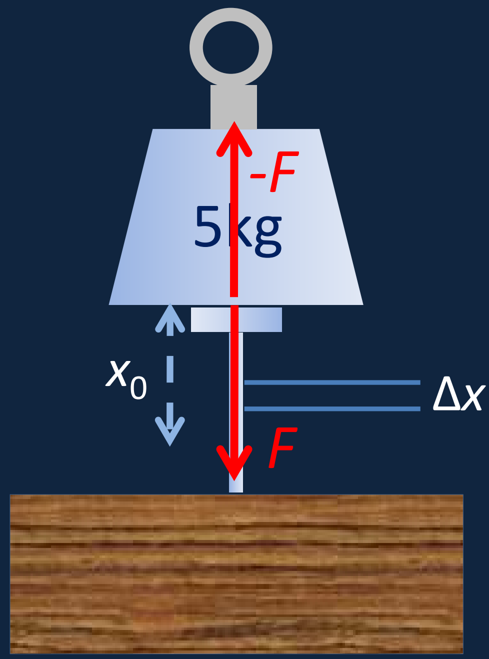
- Energy is the ability to do work: to deliver a force that acts through a distance.
- Placing a weight gently on a nail does nothing.
- Dropping the weight on the nail can drive the nail into the wood.
- If the weight is moving when it hits the nail, it has the ability to do work driving the nail in. This is its *kinetic energy.*



How Much Work Does the Moving Weight Do?

- After contact with the nail, the forces between the weight and the nail are equal and opposite. Suppose the nail is driven in a total distance x_0.
- In going through a small distance Δx, the work done on the nail $\Delta W = F\Delta x$.
- Meanwhile, for the weight $-F = ma$, the weight has slowed down: $-F = m\Delta v/\Delta t$.
- Therefore $\Delta W = F\Delta x = -m\,\Delta v\Delta x/\Delta t$.
- Now for small Δx, we can take $\Delta x/\Delta t = v$, so $\Delta W = -mv\Delta v$, and

$$W = \int_0^{x_0} dW = -\int_{v_0}^{0} mv\,dv = \tfrac{1}{2} m v_0^2$$



Weight hits nail with speed v_0

Where Did the Weight's Energy Come From?

- We've seen that if the weight hits the nail and comes rapidly to rest, it loses energy $\frac{1}{2} m v_0^2$.
- This is its kinetic energy K at speed v_0.
- Let's suppose it gained that energy by being dropped from rest at a height h.
- At uniform acceleration g, $v_0^2 = 2gh$.
- So the kinetic energy $\frac{1}{2} m v_0^2 = mgh$: precisely the potential energy lost in the fall—the work done by gravity mgh = force mg x distance h.

A Small Kinetic Energy Change

- Suppose the velocity of a mass *m* changes by a tiny amount $\Delta\vec{v}$ as the mass moves through $\Delta\vec{r}$. Then the change in kinetic energy *K* is (dropping the *very* tiny $(\Delta v)^2$ term)

$$\Delta K = \tfrac{1}{2}m\left(\vec{v}+\Delta\vec{v}\right)^2 - \tfrac{1}{2}m\vec{v}^{\,2} = m\vec{v}\cdot\Delta\vec{v} = m\frac{\Delta\vec{r}\cdot\Delta\vec{v}}{\Delta t}$$

- Note this depends only on the change in *speed*—the dot product ensures that only the component of $\Delta\vec{v}$ in the direction of $\vec{v}$ counts. The displacement $\Delta\vec{r}$ is of course in direction $\vec{v}$.

Energy Balance for a Projectile

- Consider a projectile acted on only by gravity, moving a distance $\Delta\vec{r}$ in a short time Δt.
- Gravity does work $m\vec{g}\cdot\Delta\vec{r} = -\Delta U$, where U is the gravitational potential energy.
- The change in velocity $\Delta\vec{v} = \vec{g}\Delta t$, so the change in potential energy

$$\Delta U = -m\vec{g}\cdot\Delta\vec{r} = -m\frac{\Delta\vec{v}\cdot\Delta\vec{r}}{\Delta t} = -\Delta K\,!$$

- The total energy $U + K$ does not change.

Conservation of Mechanical Energy

- We've established that for a projectile acted on only by gravity K.E. + P.E. = a constant,

$$\tfrac{1}{2}m\vec{v}^{\,2} + mgh = E,$$

- Here *E* is called the total (mechanical) energy.
- This is valid if:

A. We can neglect air resistance, friction, etc.

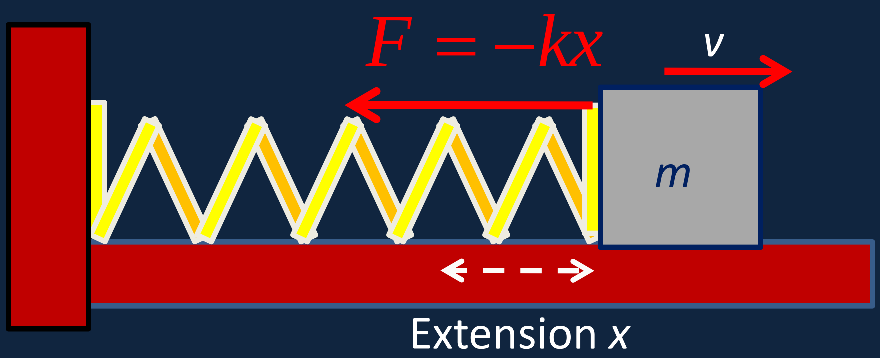
B. Other forces acting are always perpendicular to the direction of motion: so this will also be true for a roller coaster, ignoring friction.

Springs Conserve Energy, Too

- Suppose the spring is fixed to the wall, at the other end a mass *m* slides on a frictionless surface.
- By an exactly similar argument to that for gravity, we can show

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = E,$$

constant total energy.



Conservative and Nonconservative Forces

- Gravity and the spring are examples of conservative forces: if work is done against them, they store it all as potential energy, and it can be used later. Total mechanical energy is conserved.
- Friction is not a conservative force: work done against friction generates heat, it does not conserve the mechanical energy, little of which can be recovered.

Different Paths for a Conservative Force

- For a conservative force, suppose taking an object from point A to point B along path P_1 requires us to supply work W_1. Then if we let the object slide back from B to A, the force will fully reimburse us, giving back *all* the work W_1.

- Now suppose there's another path P_2 from A to B, and using *that* path takes less work from us, W_2.

- We can construct a track going from A to B along P_2 then back along P_1, and we'll gain energy! This is a perpetual motion machine…so what's wrong?

Potential Energy in a Conservative Field

- Imagine a complicated conservative field, like gravity from Earth + Moon at any point. We've established that the work we need to do to take a mass *m* from point $\vec{r}_A$ to $\vec{r}_B$,

$$W\left(\vec{r}_A, \vec{r}_B\right) = \int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r}$$

depends *only* on the endpoints, **not the path**—so we can unambiguously define a potential energy difference

$$U\left(\vec{r}_B\right) - U\left(\vec{r}_A\right) = \int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r}$$

Potential Energy Determines Force

- If we know the potential energy $U(\vec{r})$ in a complicated gravitational field, how can we find the gravitational force on a mass *m* at $\vec{r}$?
- Take a very short path going in the *x*-direction:

$$U(\vec{r}+\Delta x)-U(\vec{r})=\int_{\vec{r}}^{\vec{r}+\Delta x}\vec{F}\cdot d\vec{r}=F_x\Delta x$$

- We must apply a force F_x to move this small distance, so the opposing gravitational force is given by $F_{Gx}=-\partial U(\vec{r})/\partial x$.

More on Potential Energy and Force

- Since the potential energy is given by integrating the force through a distance, it's not surprising that we get back the force by differentiating the potential energy.

- For gravity near the Earth's surface, $U(\vec{r}) = mgz$, taking z as vertically up, so

$$F_z = -\partial U / \partial z = -mg$$

  and since U doesn't depend on x or y, there is no force in those directions.

- **Reminder!** Forces and work depend only on *changes* in potential energy—we can set the zero of potential energy wherever is convenient, like ground level.

Potential Energy and Force for a Spring

- For a spring,

$$U(x) = \tfrac{1}{2}kx^2$$

  a parabola.

The force the spring exerts when extended to *x*

$$F(x) = -dU(x)/dx = -kx$$

- It's worth staring at the $U(x)$ graph, bearing in mind that the force at any point is the *negative* of the slope there—see how it gets steeper further away from the origin.