Motion in Two and Three Dimensions: Vectors

Physics 1425 Lecture 4

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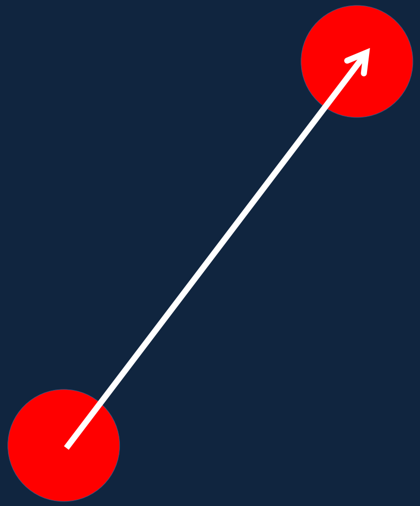
Today’s Topics

- In the previous lecture, we analyzed the motion of a particle moving vertically under gravity.
- In this lecture and the next, we’ll generalize to the case of a particle moving in two or three dimensions under gravity, like a projectile.
- First we must generalize displacement, velocity and acceleration to two and three dimensions: these generalizations are vectors.

Displacement

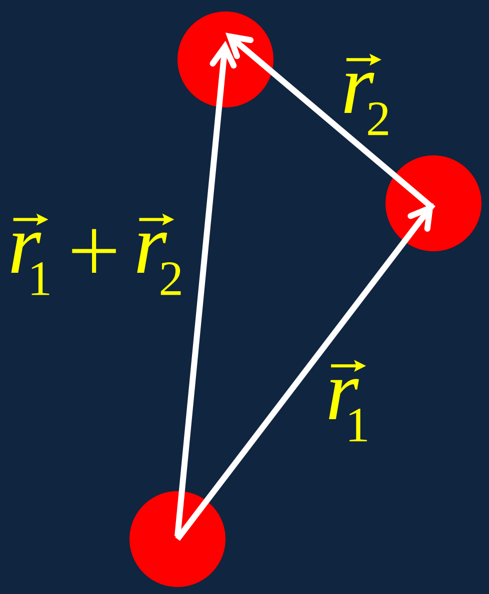
- We'll work usually in two dimensions—the three dimensional description is very similar.
- Suppose we move a ball from point A to point B on a tabletop. This displacement can be fully described by giving a distance and a direction.



- Both can be represented by an arrow, the length some agreed scale: arrow length 10 cm representing 1 m displacement, say.
- This is a vector, written with an arrow $\vec{r}$: it has magnitude, meaning its length, written $|\vec{r}|$, and direction.

Displacement as a Vector

- Now move the ball a *second* time. It is evident that the total displacement , the sum of the two, called the resultant, is given by adding the two vectors tip to tail as shown:
- Adding displacement vectors (and notation!):



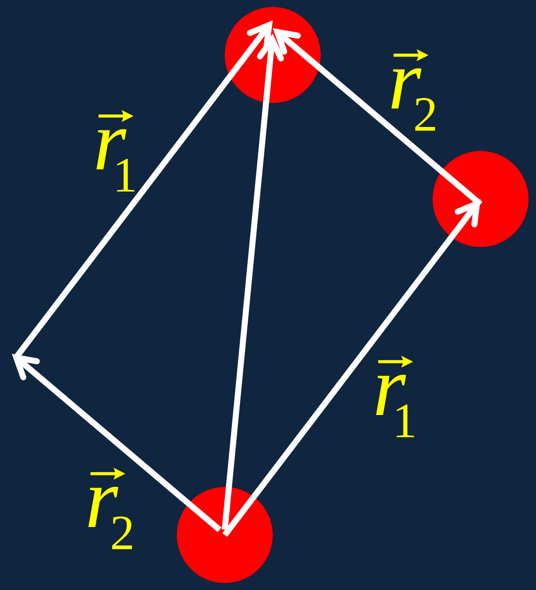
Adding Vectors

- You can see that

$$\vec{r}_1 + \vec{r}_2 = \vec{r}_2 + \vec{r}_1.$$

- The vector $\vec{r}_1$ represents a displacement, like saying walk 3 meters in a north-east direction: it works from any starting point.

- Adding vectors :



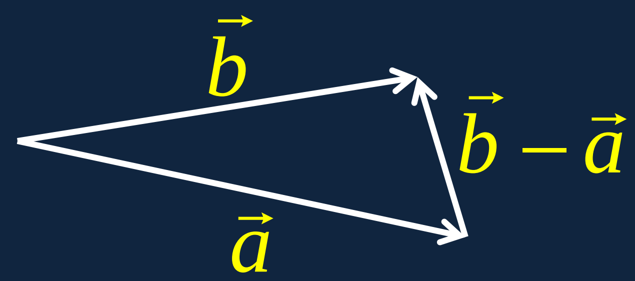
Subtracting Vectors

- It's pretty easy: just ask, what vector has to be added to $\vec{a}$ to get $\vec{b}$?
- The answer must be

$$\vec{b} - \vec{a}$$

- To construct it, put the ***tails*** of $\vec{a}$,$\vec{b}$ together, and draw the vector from the head of $\vec{a}$ to the head of $\vec{b}$.

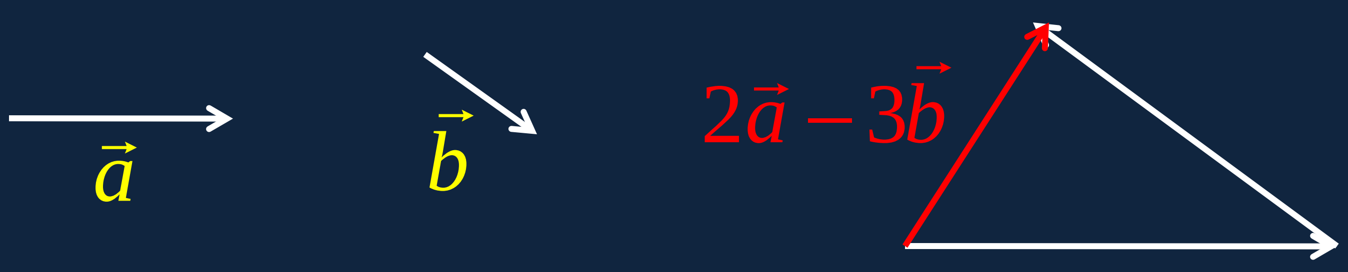
Finding the difference:



Multiplying Vectors by Numbers

- Only the length changes: the direction stays the same.

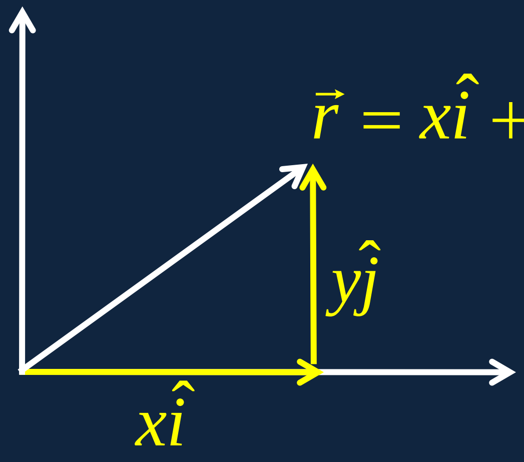
- Multiplying and adding or subtracting:



Vector Components

- Vectors can be related to the more familiar Cartesian coordinates (x, y) of a point P in a plane: suppose P is reached from the origin by a displacement $\vec{r}$.
- Then $\vec{r}$ can be written as the sum of successive displacements in the x- and y-directions:
- These are called the components of $\vec{r}$.

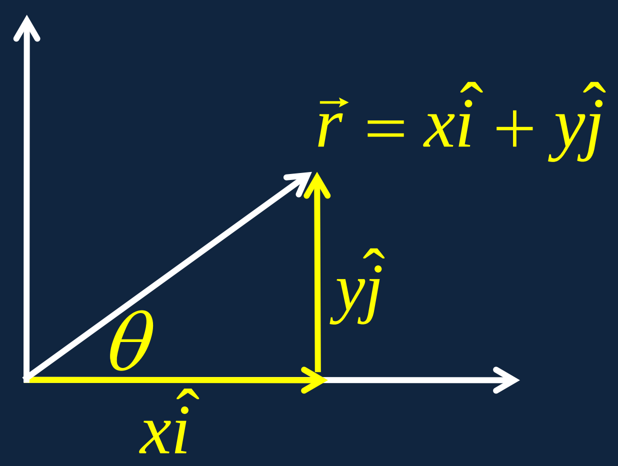
- Define $\hat{i}, \hat{j}$ to be vectors of unit length parallel to the x, y axes respectively. The components are $x\hat{i}, y\hat{j}$.



How $\vec{r}$ Relates to (x, y)

- The length (magnitude) of $\vec{r}$ is

$$\left|\vec{r}\right| = \sqrt{x^2 + y^2}$$



The angle between the vector and the x-axis is given by:

$$\tan\theta = \frac{y}{x}.$$

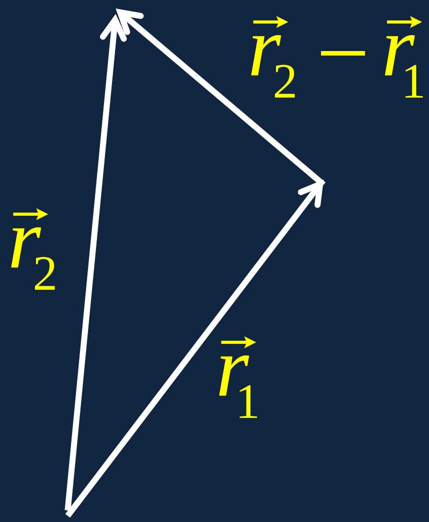
Average Velocity in Two Dimensions

average velocity = displacement/time

- A

$$\boxed{\bar{\vec{v}} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}}$$

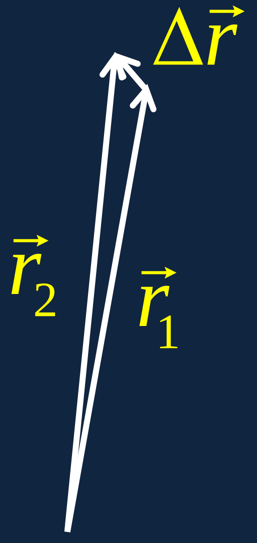
In moving from point $\vec{r}_1$ to $\vec{r}_2$, the average velocity is in the direction $\vec{r}_2 - \vec{r}_1$:



Instantaneous Velocity in Two Dimensions

$$\boxed{\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}}$$

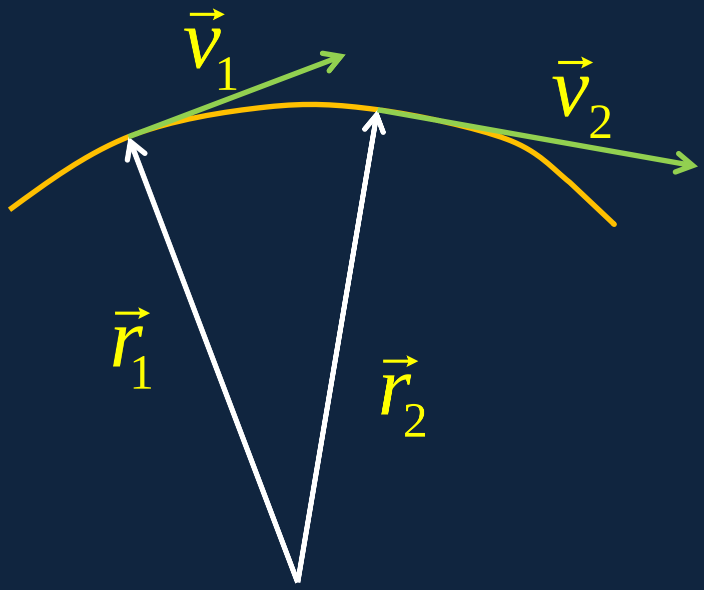
Defined as the average velocity over a vanishingly small time interval : points in direction of motion at that instant:



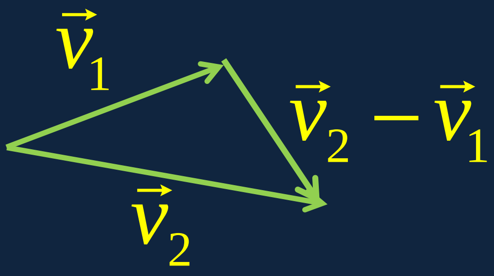
- Note: $\Delta \vec{r}$ is small, but that doesn't mean $\vec{v}$ has to be small— Δt is small too!

Average Acceleration in Two Dimensions

- Car moving along curving road:



$$\overline{\vec{a}} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$



Note that the velocity vectors ***tails*** must be together to find the difference between them.

Instantaneous Acceleration in Two Dimensions

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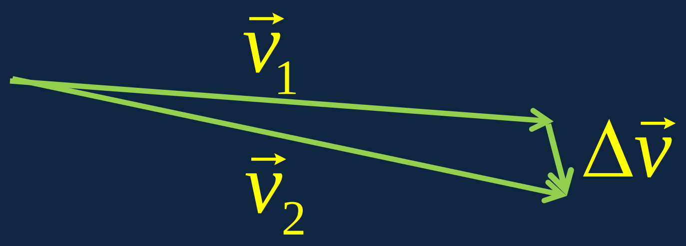


Acceleration in Vector Components

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}}$$

Writing $\vec{a} = \left(a_x, a_y\right)$, $\vec{r} = \left(x, y\right)$ and matching:

$$a_x = \frac{d^2x}{dt^2}, \quad a_y = \frac{d^2y}{dt^2}$$

as you would expect from the one-dimensional case.

Clicker Question

A car is moving around a circular track at a constant speed. What can you say about its acceleration?

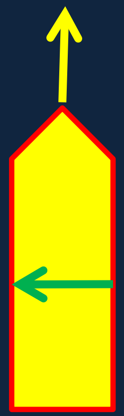
A. It’s along the track
B. It’s outwards, away from the center of the circle
C. It’s inwards
D. There is no acceleration

Relative Velocity

Running Across a Ship

- A cruise ship is going north at 4 m/s through still water.
- You jog at 3 m/s directly across

 the ship from one side to the other.

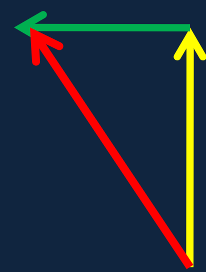


- What is your velocity *relative to the water*?

Relative Velocities Just Add…

- If the ship's velocity relative to the water is $\vec{v}_1$
- And your velocity relative to the ship is $\vec{v}_2$
- Then your velocity relative to the water is

$$\boxed{\vec{v}_1 + \vec{v}_2}$$



- Hint: think how far you are *displaced* in one second!